AN ELEMENTARY GEOMETRICAL PROPERTY OF LINKS AND KNOTS.

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Introduction

Using models of knots and of linked curves, both the following statements[1] can be confirmed in cases which are easily accessible through intuition.

(A) If there are two simple (separate) closed curves A and B which are linked in three dimensional Euclidean space, then there is always at least one straight line which cuts both curves A and B twice, in the order A, B, A, B.

(B) To every knot (lying in three dimensional Euclidean space $\mathbb{R}^3$), there is at least one straight line intersecting K four times. This is a quadrisecant.

Pairs of curves (which are unlinked in an illustrative sense) and unknotted curves may be so deformed that lines with the intersection properties from above do not exist. One must therefore regard the appearance of such a line as caused by being linked, respectively being knotted. It will be our goal in the following to obtain an exact statement about the quadrisecants of linked curves and knots. The most appropriate manner to approach these problems is the one which founds links and knots on the notion of isotopy, that is of a one-to-one, continuous deformation. Since there has not been a satisfactory treatment of one-to-one deformations so far, we have to choose from among the definitions of links and knots one which is suitable for our methods of proof. p. 630

Before we will formulate the main theorem, we will list under a) the limiting conditions which we will always make and which are mainly related to the "general position". Then we will discuss under b) in (sufficient) detail the different possibilities for the definition of links, to finally arrive (using one of these definitions) at Theorem 1. This theorem makes statement (A) above more precise. The next step under c) will be to introduce the suitable notion of knot for statement (B) and (B) will be formulated in a stronger version. In d) we will talk about generalizations.

a) In the following let us use the viewpoint of Combinatorial Topology. Curves are polygons. Here, each vertex of a polygon belongs to at most two sides. "Surfaces" (Flächenstücken) are built out of plane triangles. Each edge belongs to at most two triangles and triangles with a common vertex can be arranged (cyclically or linearly) such that each triangle appears exactly once in the ordering and two consecutive triangles have one edge in common. Edges which belong to exactly one triangle form the boundary of the surface. If the surface E is the unique simplicial image of a suitably triangulated disk (Dreieckscheibe) then E is called a "disk" (Elementarflächenstück). E is bounded by the edges of the triangles.

[1]The following work is taken from the content of my dissertation (University of Berlin 1931). I am indebted to Herrn O. Toeplitz for both remarks (A) and (B) which were the starting point for this work.

If we are given an object in $\mathbb{R}^3$, which consists of finitely many polygons and surfaces, then we will make the assumption of general position, since often the number of intersection points are to be determined. This (general position) means that not more than three vertices of the object should lie in one plane. Therefore two sides of the object can only have one point in common (it has to be a vertex) if they are either the sides of the same polygon or are sides of the triangulation of the same surface. These conditions can always be attained by arbitrarily small perturbations of vertices of the object. The condition of general position prohibits the occurrence of double points for polygons: in contrast, surfaces can have self-intersections. The number of points of intersections of a polygon with any surface is always finite, such an intersection point is always an interior point both of the edge of the polygon and of the triangles of the surface. If an edge of a polygon intersects several triangles of one surface in the same point, then the intersection point has to be counted with the corresponding multiplicity. In general, we will want absolute intersection numbers, that is, the number of intersection points without sign. If we want to count the intersection points of oriented surfaces and oriented polygons with a (predefined) sign, then we will emphasize this by using the label “algebraic
p. 631 number” of the intersection points.

If A and B are two distinct, simple closed polygons, then there are among the disks bounded by A such ones that have the smallest absolute intersection number with B. We will call this intersection number “the necessary intersection number with B for a disk bounded by A.” This notion also makes sense if A and B are identical, that is if the intersection points of A are counted with the triangles of the disks bounded by A. In this case, we will talk about the “the necessary number of boundary singularities for a disk bounded by A.”

It is known that special caution is necessary for the execution of deformations in the theory of knots and links. Let P be a simple (open or closed) polygon in $\mathbb{R}^3$, f_t $(0 \leq t \leq 1)$ a family of injective continuous map of P into $\mathbb{R}^3$ depending continuously on the parameter t. P_t, respectively $\mathcal{P}_t$, respectively p_t denote the image of the polygon P, respectively of the edges of the polygon $\mathcal{P}$, respectively of the point p under the map f_t. The family f_t $(0 \leq t \leq 1)$ is called a deformation of P if the following conditions are satisfied:

(1) $p_0 = p$ for all $p \subset P$.
(2) There is a fixed subdivision $\bar{P}$ of P such that a point p_t of P_t is a vertex of P_t if p_0 is a vertex of $\bar{P}$. The image p_t of a vertex p_0 of $\bar{P}$ is also called a vertex of polygon P_t if the point p_t is only an apparent vertex, that is if the two sides ending in p_t of P_t are in fact altogether one side.
(3) The length of polygon edge $\mathcal{P}_t$ remains for all above a fixed positive bound for all t $(0 \leq t \leq 1)$. During the deformation of P the condition of general positions can be temporarily violated, but it will be assumed to hold for the starting position P_0 and the end position P_1 - possibly after omitting apparent vertices.

A deformation of a pair of distinct simple curves A and B is a family of maps f_t, defined on A and B such that f_t defines a deformation on each of the polygons. In the following, if we talk about some deformation, we will assume that A and B are already given with a sufficient subdivision for a deformation. We say that A and B can be “separated from each other” by the deformation f_t $(0 \leq t \leq 1)$, if A_1 and B_1 can be separated from each other in $\mathbb{R}^3$ by a sphere (Kugelfläche). If

we want the intersections occurring when separating A and B we will consider the "deformations with finite intersection number". These are deformations such that: p. 632
1. For all values of t, the intersection $A_t \cap B_t$ consists of at most finitely many points. 2. $A_t \cap B_t$ is nonempty for at most finitely many values of t. A point of the intersection $A_t \cap B_t$ through which pass m edges of A_t and n edges of B_t represents a $m \cdot n$-tuple intersection of A and B under the deformation f_t.[2] The total number of points occurring in the intersections $A_t \cap B_t$ under a deformation f_t (with finite intersection number for $(0 \leq t \leq 1)$) is called the number of intersections of A and B under the deformation f_t. The intersection points should be counted with correct multiplicity. Even under restricting conditions the polygons A and B can obviously always be separated from each other through deformations with finite intersection number. By restricting conditions we mean conditions such as the exclusion of self intersections (that is the injectivity of the maps f_t onto A and onto B separately for all t) or fixing one of the polygons (example $A_t = A$ for all t.) Among all the deformations achieving the separation of A and B there are those which have the minimal number of intersections of A and B. The minimum number of intersections of A and B necessary to separate A and B is the "necessary intersection number to separate A and B."

b) The following possibilities are available for the definition of linking together two separate simple closed polygons A and B:-

Definition 1. *Link Homology.*[3] The link homology number of an oriented polygon A with the oriented polygon B is the algebraic count of the intersection points of B with an oriented surface which is orientable and bordered by A. The orientation of the surface is given by A. A is called unlinked with B when the linking number of A with B is zero, otherwise A and B are called linked.

When A and B are unlinked - and of course only then - it is known that there exists an orientable surface which is bounded by A that B is separate from. (This surface is not necessarily simply connected.) Furthermore, one knows that the link p. 633
homology is symmetric, the link number of A with B is equal to the linking number of B with A. It also makes sense to say that A and B are linked together in the sense of homology.

Definition 2. *Symmetric Link Homotopy.* The symmetric link homotopy number of A and B is that of the necessary intersection number of the separation of A and B through any deformation. A and B are called unlinked when the linking number is zero, otherwise A and B are called linked together.

It is known that two polygons which are linked together in the sense of homology, are also linked together in the sense of symmetric link homotopy. The complete equivalence of both link definitions (and therefore a connection between the homology and homotopic properties of curves in $\mathbb{R}^3$) is guaranteed by

Proposition 1. *Two polygons, which are unlinked in the sense of homology, are also unlinked in the sense of symmetric homotopy.*

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[2]If the point is a vertex of A_t or B_t, then one has to take the pair of edges with common vertex p_t instead of the edge going through p_t.

[3]The notion of link homology was introduced in more generality by L.E.J. Brouwer, 1912, Proc. Amst. **15**, p. 113-122. In this work one finds properties of link homology applied here, in particular the independence of the definition of link homology number from the choice of orientable surface bounded by A.

Proof. Section 1. □

Addition to Proposition 1. *The symmetric link homotopy number is equal in absolute value to the link homology number.*

Proof. Section 1. □

Definition 3. *Unsymmetric Link Homotopy.* The unsymmetric link homotopy number ${}_A\nu_B$ of A with B is the intersection number of A and B necessary to separate A from B using deformations while holding B fixed. Or, equivalently the necessary intersection number for a surface bounded by A with B. A is called unlinked with B when ${}_A\nu_B = 0$, otherwise A is called linked with B. The link number ${}_B\nu_A$ and B linked with A can be defined in a corresponding way.

The following holds for the unsymmetric link homotopy.

Proposition 2. *The formal unsymmetric link homotopy is actually unsymmetric.*

Proof. See Section 1 example 1. □

From the definition, it follows directly that each of the unsymmetric link homotopy numbers is equal to the symmetric link homotopy number. In particular, if A and B are linked together in the sense of symmetric homotopy then both A is linked with B as well as B with A in the sense of unsymmetric link homotopy. Meanwhile, it already follows from Proposition 2 that both definitions cannot be equivalent. It even holds that

Proposition 3. *There are pairs of curves, which are unlinked in the sense of symmetric homotopy (and therefore in the sense of homology) but are linked (one with the other and vise versa) in the sense of unsymmetric homotopy.*

Proof. Section 1 example 2. □

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The difference between the symmetric and the unsymmetric link homotopy is based on the existence of knots. If B is unknotted[4] then A is, as one knows, null homotopic in $\mathbb{R}^3 \setminus B$ if and only if A is null homologous in $\mathbb{R}^3 \setminus B$.

First Addition to Proposition 3. *If B is unknotted and if A and B are unlinked in the sense of homology or symmetric homotopy (which is the same according to Proposition 1), then A is also unlinked with B in the sense of unsymmetric link homotopy.*

More precisely,

Second Addition to Proposition 3. *If B is unknotted, then the unsymmetric link homotopy number ${}_A\nu_B$ (of A with B) is equal to the absolute value of the link homology number, also (due to the Addition to Proposition 1) it is equal to the symmetric homotopy number.*

This implies in particular that link homology, symmetric and unsymmetric link homotopy are identical for unknotted polygons A and B.

Definition 4. *Link Isotopy.* A and B are called unlinked in the sense of isotopy when the intersection number of A and B is equal to zero. This is through the separation of A and B through necessary deformations without self intersection. Otherwise A and B are called linked.

[4] Compare Definition 5 and footnote 7.

There is little point in introducing the necessary intersection number for the separation of A and B as the linking number for link isotopy as we did in the case of Definitions 2 and 3. Since, in order to describe the linking we have to take into account not only the mutual intersections of A and B, but also the self-intersections of A and B while the separation is taking place.

It is known that under the assumptions made under statement (A), each injective deformation of a polygon can be extended to an injective deformation of the whole space. This implies immediately that two polygons which are unlinked in the sense of isotopy cannot be linked with each other in the sense of unsymmetric homotopy. Indeed, if the injective deformation f_t achieves the separation of A and B without mutual intersections and if $\phi_t^{(A)}$ is an extension of the injective deformation $f_t(A)$ to the whole space and if $(\phi_t^{(A)})^{-1}$ is the inverse of $\phi_t^{(A)}$, then the map $(\phi_t^{(A)})^{-1} f_t$ is defined on A and B and is a separation of A and B without mutual intersection while fixing A. Since Definition 3 is essentially unsymmetric, but Definition 4 is symmetric, the two notions of linking are not equivalent. More precisely, p. 635

Proposition 4. *There are polygons which are linked in the sense of isotopy, but which are not linked with each other in the sense of unsymmetric homotopy.*

Proof. See Section 1 example 3. □

The invariance of Definitions 1 to 4 under injective deformation of pairs of polygons A and B or (equivalently) under injective deformations of the whole space is evident. Some proof is necessary to show the topological invariance of the linking numbers defined in 2 to 4, that is the independence of the different "necessary minimum numbers" of the geometry of $\mathbb{R}^3$.

Unsymmetric link homotopy is the most suitable notion of linking for the proof of statement (A). A "secant of the polygon pair A, B" is an oriented line which intersects the set $A+B$ in at least two points which do not lie on the same polygon edge. A line is called an "m-tuple secant" if it intersects $A+B$ in at least m-points (belonging to different polygon sides). By the "intersection sequence" of an m-tuple secant, for example $(A, B, \dots, A)$, we will understand the listing of the order of the intersection points of A and B on the oriented line. An m-tuple secant with a given intersection sequence is uniquely determined if we are given m predefined intersection points with A and B in the ordering determined by the intersection sequence. Two m-tuple secants with the same intersection sequence are identical if and only if the two oriented lines coincide and the distinguished intersection points are pairwise equal. In the following, trisecants with intersection sequence (A, B, A) and (B, A, B) and quadrisecants with the order (B, A, B, A) are important.

A strengthening of statement (A) is:-

Theorem 1. *Two pairwise distinct simple closed polygons A and B with unsymmetric link homotopy number ${}_A\nu_B$ and ${}_B\nu_A$ possess at least ${}_A\nu_B \cdot {}_B\nu_A$ quadrisecants which intersect in the sequence (B, A, B, A).*

Proof. Section 2. □

This contains the weaker, but in concrete cases more easily applicable proposition: *Two pairwise distinct simply closed polygons A and B with the link homology number u, possess at least u^2 quadrisecants which have the intersection sequence (B, A, B, A).* p. 636

The existence of trisecants with intersection sequence (A, B, A), respectively (B, A, B) for pairs of curves with unsymmetric link homotopy numbers ${}_A\nu_B$ respectively ${}_B\nu_A$, follows easily from the link definition (Section 2 Lemma 3.) The question of the existence of quadrisecants is justified by the fact that secants of higher multiplicity do not possess any invariant properties. They can be removed by an arbitrary small perturbation of the polygon edges. In fact, let the line $\mathcal{G}$ intersect the five polygonal edges s_i $(i = 1, \ldots, 5)$ of $A+B$ in five mutually distinct points. We may assume $A + B$ to be in general position. Therefore two s_i which lie in one plane will always have a common vertex, since otherwise such a plane would contain four vertices of $A + B$. Hence, $\mathcal{G}$ cannot contain any of the s_i since with one side contained in $\mathcal{G}$ each of the other four s_i would have to lie in one plane and therefore have a vertex in common. Furthermore, three s_i's cannot join to be one segment of a line, because if s_1 and s_2, s_2 and s_3 each had a vertex in common, then s_1 and s_3 would lie in the plane determined by $\mathcal{G}$ and s_2, and therefore would have to have a vertex in common. But s_1, s_2 and s_3 cannot form a triangle, since these three sides are intersected by $\mathcal{G}$ in three distinct points. Therefore there are only three possible cases for the lines S_i determined by the s_i using a suitable numbering.

(1) The S_i are pairwise skew
(2) S_1 and S_2 intersect and otherwise the S_i are pairwise skew.
(3) S_1 and S_2 intersect, S_3 and S_4 intersect, and otherwise the S_i are pairwise skew.

However, one can always choose four triples of skew lines: S_1, S_3, S_5; S_1, S_4, S_5; S_2, S_3, S_5; S_2, S_4, S_5. Each of the triples of lines determines a one-sheeted hyperboloid or a hyperbolic paraboloid $H_{\iota\kappa\lambda}$. The fact that $\mathcal{G}$ intersects all the S_i implies that the (possibly intersecting) surfaces H_{135}, H_{145}, H_{235}, H_{245} have the line $\mathcal{G}$ as a common generator. By a small perturbation of the vertices of s_i one can show that H_{135} and H_{235} are distinct. Therefore H_{135} and H_{235} have only the generators S_3 and S_5 in common in one of the families of generators, hence they have two generators $\mathcal{G}'$ and $\mathcal{G}''$ in common in the other family of generators.[5] By a small perturbation of s_4, one can show that S_4 is distinct from $\mathcal{G}'$ and $\mathcal{G}''$. After doing
p. 637 these two alterations, there is no line left which intersects all the S_i. By making the perturbations of the vertices small enough, one can prevent the occurrence of new quintisecants or multiple-secants on the sides of A and B which are distinct from the s_i's.

c) We will base the proof of statement (B) on the following definition of knots, which will avoid the notion of injective deformation.

Definition 5. The knotting number k of a simple closed polygon K is the necessary number of boundary singularities for a disk bounded by K.[6]

K is called unknotted if $k = 0$ otherwise K is called unknotted.[7]

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[5]One of the two generators can be an improper (uneigentliche) line.

[6]Compare with definition under (A).

[7]After M.Dehn, Math Annalen **69** (1910), p. 137-168, this definition is equivalent to the following: K is called unknotted if and only if the fundamental group of $\mathbb{R}^3 \setminus K$ is commutative, and then it is necessarily the free abelian group on one generator. In contrast, whether (by Definition 5) an unknotted K is always isotopic to a circle has to remain unproved. This is because of a gap in the proof of the Dehn Lemma which was brought to light by H. Kneser, Jahresber. D.M.V. **38** (1929) p. 248.

The knotting number k is obviously invariant under injective deformations of K and under injective deformations of $\mathbb{R}^3$. We can say the same about the topological invariants as in the case of linking numbers. The following holds;

Proposition 5. *k is always an even number; therefore for knotted polygons $k \geq 2$.*

Proof. See Section 3. □

By an m-tuple secant $m > 2$ of K, we will mean a line (this time not with an orientation) which intersects K in at least m points, of which no two belong to the same polygon edge. An m-tuple secant is only then uniquely characterized if the m-intersection points with K are distinguished on it. Two m-tuple secants are identical if and only if both lines coincide and the distinguished intersections points with K agree pairwise. As in the case of links, the knot definition easily implies the existence of trisecants (Section 4 Lemma 11). Quintisecants and multi-secants can again be removed. The following hold for quadrisecants;

Theorem 2. *A simple closed polygon K in $\mathbb{R}^3$ with the knotting number k possesses at least $k^2/2$ quadrisecants.*

Proof. Section 4. □

Strengthening statement (B), yields the following: *Each knot possesses at least two quadrisecants.*

d) Let us make some remarks about the question of how far the stated results can be improved. For links there are simple examples for which the lower bound given in Theorem 1 for the number of quadrisecants with intersection sequence (B, A, B, A) is attained - for example two triangles linked with each other. But the bound is zero for a pair of curves which are only linked in the sense of isotopy (compare Section 1 example 3), while simple examples seem to confirm, also in this case, the existence of quadrisecants. - Whether the bound of Theorem 2 is attained p. 638
in the case of knots has to remain undecided, even in the simple case of the clover leaf knot ($k = 2$).

The condition of general position of polygons is helpful for the execution of the proofs, but is not essential. As one easily sees, one can always attain general position of a pair of polygons, respectively of a knot by a small perturbation of the vertices without increasing the number of quadrisecants. This implies the validity of the two main theorems also for polygons which do not satisfy the condition of general positions.

Statement (A) (contained in Theorem 1) for polygons easily leads to the proof of statement (A) for arbitrary continuous curves using a suitable definition of linking (without the strengthening done by Theorem 1.) Let A and B be two distinct simple closed curves in $\mathbb{R}^3$. A and B are called linked (with each other) if there exists an $\epsilon > 0$, such that for each pair of polygons A_ϵ, B_ϵ which approximate A, B up to ϵ we have ${}_{A_\epsilon}\nu_{B_\epsilon} \neq 0$ as well as ${}_{B_\epsilon}\nu_{A_\epsilon} \neq 0$. For a linked pair of curves A, B each pair of approximating polygons A_ϵ, B_ϵ possess a quadrisecant with intersection sequence $(B_\epsilon, A_\epsilon, B_\epsilon, A_\epsilon)$. If $A^{(\nu)}$, $B^{(\nu)}$ is a sequence of approximating polygons converging to A, B, then there exists at least one line G to which the quadrisecants cluster with the intersection sequence $(B^{(\nu)}, A^{(\nu)}, B^{(\nu)}, A^{(\nu)})$. G is then obviously a quadrisecant of A, B with intersection sequence (B, A, B, A).

Theorem 1 remains true for continuous curves if one uses the following definition of linking and knotting number. Let ${}_A\nu_B^{(\epsilon)}$ and ${}_B\nu_A^{(\epsilon)}$ be the minimum number

among the ${}_{A_\epsilon}\nu_{B_\epsilon}$ and ${}_{B_\epsilon}\nu_{A_\epsilon}$ for all polygon pairs A_ϵ, B_ϵ approximating the pair of curves A, B up to ϵ. Let $k^{(\epsilon)}$ be the respective minimum number among the knotting numbers of the polygons approximating the double point free curve K up to ϵ. For ϵ tending towards zero, the numbers ${}_A\nu_B^{(\epsilon)}$, ${}_B\nu_A^{(\epsilon)}$ and $k^{(\epsilon)}$ do not decrease. If these numbers finally remain finite and constant for $\epsilon \to 0$, then this finite number shall be the number of ${}_A\nu_B$ respectively ${}_B\nu_A$, respectively k, otherwise ${}_A\nu_B$
p. 639 respectively ${}_B\nu_A$, respectively k will be infinite.[8] The proof of the main theorems for continuous curves will not be dealt with in this work, since it requires lengthy approximations. The difficulty consists in showing that the quadrisecants, which exist in each approximation in the required number, converge to distinct secants of the polygon pairs, respectively of the knots.

1. Notions of Links. Proofs of Propositions 1-4.

In the proof of Proposition 1 we require two simple lemmas about the resolution of knots with self-intersections.

Let **K** be a closed polygon in the plane. Let **K** be regular, that is the vertices of **K** should not be double points of **K** and exactly two edges of **K** intersect in each double point. **K** is called labeled (normiert) if at each double point, one of the two edges is labeled an "over-crossing" and the other is labeled an "under-crossing". We can apply the alteration allowed under knot projections to the regular labeled polygon **K**. Besides being injective deformations $\mathbf{K}_t$ $(0 \le t \le 1)$ of **K** under which vertices of $\mathbf{K}_t$ are double points of $\mathbf{K}_t$ only for finitely many t values, the alterations consist of the following three moves.

(1) Creation of a loop (Ω_1) or resolution of a loop (Ω_1') (Fig 1a).
(2) Creation of two neighboring overpasses (or underpasses) (Ω_2) and resolution of two neighboring overpasses (or underpasses) (Ω_2') (Fig 1b).
(3) The sliding of one side (lying over an intersecting pair of sides) over the intersection point of the pair of sides (Ω_3) (Fig 1c).[9]

The labeled polygon **K** is called "completely reducible" if it can be transformed to a simple closed polygon by moves of the described type.

Lemma 1. *Each regular polygon can be labeled in such a way that it is completely reducible.*

Proof. A regular polygon with at most two double points and any labeling can be transformed into an simple closed polygon by at most two applications of operations
p. 640 (Ω_1') and is therefore always completely reducible. Let us therefore assume the lemma to be true for polygons with at most $(n-1)$ double points. Let **K** be a regular polygon with n double points. Each double point divides **K** into two closed sub-polygons. Choose a double point π such that one of the two sub-polygons - call it $\mathbf{K}'$ - possesses at minimal number of sides. $\mathbf{K}'$ is then double point free. The

[8]The example of the Antoine curve A and a circle in $\mathbb{R}^3 \setminus A$ which does not bound any disk shows, that ${}_B\nu_A$ can be equal to zero even though B cannot contract to a point in $\mathbb{R}^3 \setminus A$. Therefore the here defined linking numbers describes the linking of continuous curves only very imperfectly. Compare, L. Antoine, Journ. de Math. (4) **8** (1921) p. 311-325; J.W. Alexander, Proc. U.S.A. Acad. **10** (1924), p. 8-10.

[9]The labels (Ω_i) are chosen according to K.Reidemeister's "Knotentheorie", Ergebnisse der Mathematik und ihrer Grezgebiete **1**, Heft 1 (1932): For the details, one should refer to this report. Compare also to K. Reidemeister, Hamb. Abh. **5** (1926), p. 24-32.

other sub-polygon $\mathbf{K}''$, for which π is not a double point any more, has at most $(n-1)$ double points. Chose a labeling for $\mathbf{K}''$ which makes it completely reducible. $\mathbf{K}$ will be labeled in the following way: at the point π the choice of "overpass" or "underpass" is random. For the double points of $\mathbf{K}$ which are also double points of $\mathbf{K}''$ we will choose the labeling of $\mathbf{K}''$ for $\mathbf{K}$ as well. For the double points of $\mathbf{K}$ which are intersection points of $\mathbf{K}'$ with $\mathbf{K}''$, we will call the side belonging to $\mathbf{K}'$ the overpass. Indeed, using this labeling, $\mathbf{K}$ is completely reducible. Of the two sides of $\mathbf{K}$ which intersect in π, one must belong to $\mathbf{K}'$ and the other to $\mathbf{K}''$. One can determine two points ρ and σ on the two parts of the line belonging to $\mathbf{K}'$ such that the two segments $\pi\rho$ and $\pi\sigma$ do not contain any other double point of $\mathbf{K}$ besides π and such that the segment $\rho\sigma$ is distinct from $\mathbf{K}$ except for the endpoints. Let Λ' be the part of $\mathbf{K}'$ connecting ρ with σ and which does not contain π. Λ' together with the segment $\rho\sigma$ forms a simple closed polygon. Therefore, there exists a deformation f_t $(0 \leq t \leq 1)$ of Λ' which transforms the segment Λ' into the segment $\rho\sigma$ while fixing the endpoints and such that

(1) Λ'_t is double point free.
(2) Λ'_t is distinct from $\rho\sigma$ except for the endpoints for $t \neq 1$. Furthermore, one can also attain, using a small modification of f_t if necessary, that
(3) The intersection of Λ'_t with $\rho\pi + \mathbf{K}'' + \pi\sigma$ consists of finitely many points for all t and that it (except for ρ and σ) contains vertices of Λ'_t or $\rho\pi + \mathbf{K}'' + \pi\sigma$ only for finitely many t values.

f_t is extended to a deformation of $\mathbf{K}$ in the following way: for every point ξ of $\rho\pi + \mathbf{K}'' + \pi\sigma$ we set $f_t(\xi) = \xi$ for all t. Under the deformation f_t, double points of $\mathbf{K}_t$ can only be created or destroyed if either a vertex of Λ'_t crosses a side of $\rho\pi + \mathbf{K}'' + \pi\sigma$ or a side of Λ'_t crosses a vertex of $\rho\pi + \mathbf{K}'' + \pi\sigma$. In both cases, pairs of neighboring double points are created or destroyed. Let us make the convention that at each double point created under the deformation, the side belonging to Λ'_t is an overpass of the side belonging to $\rho\pi + \mathbf{K}'' + \pi\sigma$. Then the deformation f_t gives an allowed move of the labeled polygon $\mathbf{K}$, since each birth or death of a double point can be interpreted as an operation Ω_2, respectively Ω'_2, and each move of one side of Λ'_t over a double point of $\rho\pi + \mathbf{K}'' + \pi\sigma$ can be interpreted as an operation Ω_3. The resulting $\mathbf{K}_1$ of the moves of $\mathbf{K}$ defined in the above way, the polygon p. 641
$\mathbf{K}'' + \pi\rho + \rho\sigma + \sigma\pi$ has a double π, which can be removed by an operation Ω'_1. Doing this transforms $\mathbf{K}_1$ into the polygon $\mathbf{K}''$, which is completely reducible by our assumption. □

If K is a simple closed polygon in $\mathbb{R}^3$, then the orthogonal projection of K onto a plane E is a regular polygon $\mathbf{K}$ - except for certain singular directions of projection - which is labeled in a natural way the the position of K in space. It is well known that the injective deformations of K with the precautions mentioned in the introduction correspond to the allowed moves of the projections $\mathbf{K}$ labeled in the natural way and vise versa. Labeled polygons in the plane which can be transformed into each other through allowed moves, are also projection of polygons in space representing the same knot. In particular, the completely reducible labeled polygons are projection of such unknotted polygons which bound disks without singularities. [10]

[10] A polygon K with completely reducible projection is obtained though a continuous injective deformation from a polygon K^* with double point free projection. K^* bounds a disk E^* without singularities and if we extend the deformation of K^* to an injective deformation of the whole space, then E^* is transformed into a disk E which is bounded by K.

Lemma 2. *Let A be a simple closed polygon in $\mathbb{R}^3$. Let $\mathcal{G}$ be the fundamental group of $\mathbb{R}^3 \setminus A$. Among the curves of $\mathbb{R}^3 \setminus A$ which represent a given element $\mathfrak{g}$ of $\mathcal{G}$, there is always a simple closed unknotted polygon with completely reducible projection.*

Proof. First, one can always choose a double-point free polygon G as a representative of the group element $\mathfrak{g}$. Let $\mathbf{A}$ and $\mathbf{\Gamma}$ be orthogonal projections of A and G in a plane E. E is chosen in such a way that the condition of regularity of the projection is satisfied for all double points of $\mathbf{A} + \mathbf{\Gamma}$. Let $\mathbf{\Gamma}_0$ be the polygon $\mathbf{\Gamma}$ in the labeling determined by the position of G in space. Let $\mathbf{\Gamma}_1$ be a labeling of $\mathbf{\Gamma}$ which makes $\mathbf{\Gamma}$ completely reducible - this exists according to Lemma 1. G can be deformed into a double-point free polygon G_1 on the "cylinder" with self-intersections in $\mathbb{R}^3 \setminus A$ formed by the rays of the projection, such that the projection of G_1 onto E with the labeling given by its position in space is the polygon $\mathbf{\Gamma}_1$. G_1 is then a representative
p. 642 of $\mathfrak{g}$ with the required properties. The deformation of G into a polygon G_1 is done according to the following rule: let $\pi^{(\chi)}$ $(\chi = 1, 2, \ldots, k)$ be the double points of $\mathbf{\Gamma}$ in which the labeling of $\mathbf{\Gamma}_0$ and $\mathbf{\Gamma}_1$ are distinct. On each of the underpasses of $\pi^{(\chi)}$ under the labeling of $\mathbf{\Gamma}_0$, one chooses four points $\mu^{(\chi)}$, $\nu^{(\chi)}$, $\rho^{(\chi)}$, $\sigma^{(\chi)}$ which, together with $\pi^{(\chi)}$, lie in the sequence $\mu^{(\chi)}$, $\nu^{(\chi)}$, $\pi^{(\chi)}$, $\rho^{(\chi)}$, $\sigma^{(\chi)}$. The four points are chosen close enough to $\pi^{(\chi)}$ so that the segments $\mu^{(\chi)}\sigma^{(\chi)}$ do not contain any intersection point of $\mathbf{\Gamma}$ with $\mathbf{A}$ and do not contain any double point of $\mathbf{\Gamma}$ (except for $\pi^{(\chi)}$) and such that two segments belonging to different χ are distinct. Let $m_0^{(\chi)}$, $n_0^{(\chi)}$, $r_0^{(\chi)}$, $s_0^{(\chi)}$ be the points of G which project onto the points $\mu^{(\chi)}$, $\nu^{(\chi)}$, $\rho^{(\chi)}$, $\sigma^{(\chi)}$. These four points are to be regarded as vertices of $G = G_0$. Let f_t $(0 \le t \le 1)$ be the following deformation of G_0

(1) For each vertex e_0 of G_0 distinct from $n_0^{(\chi)}$ and $r_0^{(\chi)}$, let $e_t = e_0$ for all t.
(2) If e_0 is one of the vertices $n_0^{(\chi)}$ or $r_0^{(\chi)}$, then let e_t be a point which lies on the ray of projection through e_0 with distance $\lambda \cdot t$ above e_0 (the constant factor λ will be chosen below).
(3) Each side of $e_0'e_0''$ of G_0 is proportionally mapped to the corresponding side $e_t'e_t''$ of G_t.

Choose λ large enough, such that for all χ, there exists a value $t^{(\chi)}$ $(0 < t^{(\chi)} < 1)$ and an intersection point of the side $n_{t^{(\chi)}}^{(\chi)} r_{t^{(\chi)}}^{(\chi)}$ with the side of G_0 which projects onto the side which is the overpass at $\pi^{(\chi)}$ under the labeling $\mathbf{\Gamma}_0$. Then, the labelings of the projections of G_t for all $t > t^{(\chi)}$ given by the position of G_t in space, coincides with the labeling in $\mathbf{\Gamma}_1$. Therefore the projection of G_1, labeled by the position in space of G_1, coincides with $\mathbf{\Gamma}_1$ and is thus completely reducible. The whole deformation takes place in $\mathbb{R}^3 \setminus A$. Since under the definition of f_t only such points of the space are moved through by G_t which either belong to G_t itself or lie on the rays of projection going through the segments $\mu^{(\chi)}\sigma^{(\chi)}$, but the segments $\mu^{(\chi)}\sigma^{(\chi)}$ are distinct from the projection of A. □

Proof of Proposition 1: Let A and B be two polygons unlinked in the sense of homology. By Lemma 2 we can deform B in $\mathbb{R}^3 \setminus A$ into an unknotted polygon B_1. Since B is homologous to zero in $\mathbb{R}^3 \setminus A$, B_1 is also homologous to zero in $\mathbb{R}^3 \setminus A$ and therefore A is homologous to zero in $\mathbb{R}^3 \setminus B$. Furthermore, since B_1 is unknotted, the fundamental group and homology class group of $\mathbb{R}^3 \setminus B_1$ are identical. A is also homotopic to zero in $\mathbb{R}^3 \setminus B_1$, that is A bounds a disk E in $\mathbb{R}^3 \setminus B_1$. On E we can deform A into a polygon A_1 which is totally contained in a ball of prescribed radius around an arbitrary point of E. If one chooses this ball so small that B_1 is contained in its complement, then with the polygons A_1 and B_1, we have achieved a separation of A and B. p. 643

The proofs of the Addition to Proposition 1 and the Second Addition to Proposition 3 can be done together. For that, let it be remarked that the conditions of the Second Addition to Proposition 3, which says that B is unknotted, can also be made without loss of generality for the proof of the Addition to Proposition 1. By Lemma 2 we can deform B (if it is still knotted) into an unknotted polygon in $\mathbb{R}^3 \setminus A$. (The deformation can have self-intersections.) This does not change the link homology number and the symmetric homotopy link number. In the following let us therefore assume that B is unknotted. Let u be the homology link number, v be the symmetric homotopy link number and let ${}_A\nu_B$ be the unsymmetric homotopy link number of A with B. The definition of v and ${}_A\nu_B$ implies that $v \leq {}_A\nu_B$ (see Introduction). Furthermore $|u| \leq v$ also holds for the following reason. If f_t $(0 \leq t \leq 1)$ is a deformation of the pair of polygons A,B under which exactly one intersection of A and B occurs, then the value of u for $t = 1$ can differ from the value of u for $t = 0$ by at most ± 1.[11] If one performs a deformation f_t $(0 \leq t \leq 1)$ on A and B which achieves the separation of A and B with exactly v intersections of A and B, then the values of u for $t = 0$ and $t = 1$ differ by at most $|v|$. Since A_1 and B_1 are separated from each other, we have $u = 0$ for $t = 1$ and therefore we had $|u| \leq v$ for $t = 0$.

In order to deduce the equality $|u| = v = {}_A\nu_B$ from the inequality $|u| \leq v \leq {}_A\nu_B$, we will show the inequality $|u| \geq {}_A\nu_B$. For that we have to show the following: there exists a disk bounded by A which B intersects in at most $|u|$ points. Let E be a disk bounded by A. The algebraic count of the intersection points of E with B equals u. If the number of intersection points of E with B, counted without sign, is greater than $|u|$, then one can replace E by a disk E' which has two points of intersection less than E with B. Since $|u|$ is not equal to the absolute count of intersection points of B with E, then there exists intersection points of B with E with both signs. Let p_1 and p_2 be a pair of intersection points of E with B with different signs. Let $\mathcal{E}$ be an "abstract original" of E; the points p_1 and p_2 correspond to two points $\mathfrak{p}_1$ and $\mathfrak{p}_2$ in $\mathcal{E}$. Besides those two points, there exists a finite number of points $\mathfrak{q}_i$ in $\mathcal{E}$ which are original points of the intersection points of B with E which are distinct from p_1 and p_2. One constructs a simple closed curve $\mathcal{R}$ in $\mathcal{E}$ containing the points $\mathfrak{p}_1$ and $\mathfrak{p}_2$ but excluding the points $\mathfrak{q}_i$. Let $\mathfrak{e}$ be p. 644
the sub-disk of $\mathcal{E}$ bounded by $\mathcal{R}$. To the curve $\mathcal{R}$ and the element of $\mathfrak{e}$ bounded by it in $\mathbb{R}^3$, there corresponds a curve K and a sub-disk e of E bounded by K. Since the element e has only the two intersection points p_1 and p_2 with B, and these two points have different signs, then K is homologous to zero in $\mathbb{R}^3 \setminus B$. Since B is unknotted, K is homotopic to zero in $\mathbb{R}^3 \setminus B$, therefore K bounds a disk E' which is distinct from B. If one now omits the disk e from E and instead glues the disk e' with boundary curve K, then one obtains a disk E' which possess two intersection points with B less than E. This method can be repeated as long as there are intersection points with B of different signs. If there are no more such intersection points then the absolute count of the intersection points of the disk with B is equal to $|u|$ and thus we have attained the goal of our construction.

[11] Compare L.E.J. Brouwer loc. cit.

Proof of Proposition 2 *Example 1* (Fig. 2). Let A be a knot. The fundamental group of $\mathbb{R}^3 \setminus A$ is non-commutative, it therefore contains a commutator different from the identity. Such a commutator can be - according to Lemma 2 - represented by an unknotted polygon B. B is not homotopic to zero in $\mathbb{R}^3 \setminus A$, hence ${}_B\nu_A \neq 0$. However, in contrast, B is homologous to zero in $\mathbb{R}^3 \setminus A$ and A is homologous to zero in $\mathbb{R}^3 \setminus B$. Since B is unknotted, A is homotopic to zero in $\mathbb{R}^3 \setminus B$, that is ${}_A\nu_B = 0$.

Proof of Proposition 3 *Example 2* (Fig. 3). Let A be a knot. A parallel curve of A will be a curve which together with A bounds a two-edged strip without singularities (the topological image of annulus.) Each parallel curve of A represents the same type of knot as A. There exists a parallel curve of A which is homologous to zero in $\mathbb{R}^3 \setminus A$.[12] Let B be such a curve. Since B is homologous to zero in $\mathbb{R}^3 \setminus A$, A and B are unlinked in the sense of homology. Therefore - according to Proposition 1 - they are unlinked in the sense of symmetric homotopy. But B is not homotopic to zero in $\mathbb{R}^3 \setminus A$. Since, if B bounded a disk E in $\mathbb{R}^3 \setminus A$, then one could obtain a disk bounded by A without boundary singularities by gluing E to the boundary curve B of the singularity free strip bounded by B and A. This is impossible since
p. 645 A is unknotted. Therefore we have ${}_B\nu_A \neq 0$ and - since these arguments hold if we interchange A and B - ${}_A\nu_B \neq 0$ as well.

Proof of Proposition 4 *Example 3.* For the pair of curves A, B shown in Figure 4, L. Antoine[13] showed for the first time, that A and B cannot be separated from each other by an injective deformation, that is, that they are linked in the sense of isotopy. It is evident that B can be contracted (with a self intersection) to a point in $\mathbb{R}^3 \setminus A$. Therefore B is not linked with A in the sense of unsymmetric homotopy. Since B is unknotted, it follows from the Addition to Proposition 3 that A is not linked with B in the sense of unsymmetric homotopy.

2. Multi-secants of linked pairs of polygons. Proof of Theorem 1.

The proof of the Theorem about the number of quadrisecants of polygon pairs A, B is based on the fact that quadrisecants with intersection sequence (B, A, B, A) can be interpreted as those lines on which the first pair of intersection points of a trisecant with intersection sequence (A, B, A) coincides with the last pair of intersection points of a trisecant with intersection sequence (B, A, B). The following lemmas about trisecants with intersection sequence (A, B, A) also hold for trisecants with intersection sequence (B, A, B) by interchanging A and B.

Lemma 3. [14] *Each point of polygon A is the starting point of at least* ${}_A\nu_B$ *trisecants with intersection sequence* (A, B, A).

Proof. Let there be r ($r \geq 0$) trisecants with the intersection sequence (A, B, A) among the secants with intersection sequence (A, A) and fixed starting point a_0. If a runs through the polygon A once, then the segments a_0a sweep out a disk bounded by A which is intersected by B in exactly r points. Therefore $r \geq_A \nu_B$. □

p. 646

[12] M.Dehn l.c.

[13] L.c. p. 290 - 294. The proof can be simplified by showing the following: The fundamental group of $\mathbb{R}^3 \setminus (A + B)$ is not the free group on two generators.

[14] This lemma is not necessary for the proof of the theorem. It shall only confirm the existence of trisecants and it shall give a clear picture of the set of these secants.

Let $a_0 b_0 \bar{a}_0$ be a trisecant with intersection sequence (A, B, A). An ϵ-neighborhood of the trisecant $a_0 b_0 \bar{a}_0$ in the set of secants with intersection sequence (A, B, A) is understood to be the collection of all trisecants $ab\bar{a}$ for which the distances aa_0, bb_0, $\bar{a}\bar{a}_0$ measured on the polygons A respectively B are all less than ϵ. It suffices to consider sufficiently small neighborhoods, which do not contain the full "run" through a polygon and in which the neighborhoods of the points a_0 and $\bar{a}_0$ are distinct from each other. Through the notion of neighborhood, the notion of limit in the set of trisecants with intersection sequence (A, B, A) is determined.

By an "elementary part" (Elementarbestandteil) $E_{\mathfrak{a}\mathfrak{b}\bar{\mathfrak{a}}}$ (determined by the edges $\mathfrak{a}$, $\mathfrak{b}$, $\bar{\mathfrak{a}}$ of the polygon pair A, B) of the set of trisecants with intersection sequence (A, B, A), we will understand the collection of trisecants with this intersection sequence having intersection points with A and B belonging to the sides $\mathfrak{a}$, $\mathfrak{b}$, $\bar{\mathfrak{a}}$ in this order. A trisecant of $E_{\mathfrak{a}\mathfrak{b}\bar{\mathfrak{a}}}$ which contains at least one vertex among its intersection points with $\mathfrak{a}$, $\mathfrak{b}$, $\bar{\mathfrak{a}}$ will be called a vertex trisecant. All the other trisecants will be called inner trisecants of the elementary part.

Lemma 4. *An elementary part of the set of trisecants with intersection sequence (A, B, A) is either empty or homeomorphic to a closed interval. Each non-empty elementary part possesses exactly two vertex trisecants, these correspond to the ends of the intervals. The intersection of two different elementary parts is either empty or it consists of exactly one vertex trisecant. Each vertex trisecant belongs to exactly two elementary parts.*

Proof. The lines $\mathcal{A}$, $\mathcal{B}$, $\bar{\mathcal{A}}$ are determined by the edges $\mathfrak{a}$, $\mathfrak{b}$, $\bar{\mathfrak{a}}$. Because of the general position of $A+B$ these are either distinct, or $\mathcal{A}$ and $\bar{\mathcal{A}}$ have a vertex of A in common and B intersects the plane determined by $\mathcal{A}$ and $\bar{\mathcal{A}}$. In the first case, the lines which meet $\mathcal{A}$, $\mathcal{B}$, $\bar{\mathcal{A}}$ form a system of generators of E of a one-sheeted hyperboloid or hyperbolic paraboloid, therefore they are a one-parameter family of lines. One may assume that the intersection points with $\mathcal{A}$, $\mathcal{B}$, $\bar{\mathcal{A}}$ lie in the right order on the lines belonging to E - otherwise $E_{\mathfrak{a}\mathfrak{b}\bar{\mathfrak{a}}}$ is empty. In the second case, the collection of lines which intersects $\mathcal{A}$, $\mathcal{B}$, $\bar{\mathcal{A}}$ consists of a family of lines determined by the plane $\mathcal{A}\bar{\mathcal{A}}$ and the plane's intersection point with $\mathcal{B}$. The parallel lines to $\mathcal{A}$ and $\bar{\mathcal{A}}$ have to be omitted in this family, hence the family of (oriented) lines falls in to four parts. Let E be the (uniquely determined) part of this family whose generators intersect $\mathcal{A}$, $\mathcal{B}$, $\bar{\mathcal{A}}$ in the correct order. In both cases let $E_{\mathfrak{a}}$, $E_{\mathfrak{b}}$, $E_{\bar{\mathfrak{a}}}$, be the collection of those lines of E which intersect $\mathcal{A}$, respectively $\mathcal{B}$, respectively $\bar{\mathcal{A}}$ in the points of $\mathfrak{a}$, respectively $\mathfrak{b}$, respectively $\bar{\mathfrak{a}}$. $E_{\mathfrak{a}}$ and $E_{\bar{\mathfrak{a}}}$ are always homeomorphic and $E_{\mathfrak{b}}$ is in p. 647
general homeomorphic to a closed interval. Only in the case of the family of lines is $E_{\mathfrak{b}}$ all of E. In each case the intersection $E_{\mathfrak{a}\mathfrak{b}\bar{\mathfrak{a}}}$ of $E_{\mathfrak{a}}$, $E_{\mathfrak{b}}$ and $E_{\bar{\mathfrak{a}}}$ is homeomorphic to the intersection of three or two closed intervals. It is therefore either empty or homeomorphic to a closed interval or it consists of only one line. But the last case cannot occur. The lines in $E_{\mathfrak{a}}$, $E_{\mathfrak{b}}$, $E_{\bar{\mathfrak{a}}}$ which correspond to the endpoints of the intervals are precisely the lines going through the endpoints of $\mathfrak{a}$ respectively, $\mathfrak{b}$ respectively, $\bar{\mathfrak{a}}$. The elementary part $E_{\mathfrak{a}\mathfrak{b}\bar{\mathfrak{a}}}$ is therefore always bounded by vertex trisecants. If $E_{\mathfrak{a}\mathfrak{b}\bar{\mathfrak{a}}}$ consists of one line then this line has to contain vertices of at least two of the three segments $\mathfrak{a}$, $\mathfrak{b}$, $\bar{\mathfrak{a}}$. This is impossible because of the general position of A and B, since in this case the plane determined by this line and the third side of the polygon would contain four vertices of the pair or polygons A, B.

Let $ab\bar{a}$ be a trisecant with intersection sequence (A, B, A). If the points a, b, $\bar{a}$ are interior points of a polygon edge, then the elementary part, which these trisecants belong to, is uniquely determined by the polygon edges going through a, b, $\bar{a}$. Otherwise, only one of the three points a, b, $\bar{a}$ is a vertex as we have just shown. Therefore exactly two elementary parts can be constructed so that the trisecant $ab\bar{a}$ belongs to them, namely as a vertex trisecant. □

One can regard the set of trisecants with intersection sequence (A, B, A) as a one-dimensional complex according to Lemma 4; the vertex trisecants are the vertices, the non-empty elementary parts form the edges of the complex. Since there are only a finite number of polygon edges, there is also only a finite number of elementary parts and the complex is finite. Since, by Lemma 4, each vertex belongs to exactly two edges, the creation of this complex is determined.

The following holds:

Corollary to Lemma 4. *The set of trisecants with intersection sequence (A, B, A) is either empty or it is homeomorphic to a (not necessarily connected and closed) one-dimensional complex, whose components are simply closed polygons.*

Lemma 5. *On at least one of the two polygons A, B there is a point, which is not the middle intersection point of a trisecant with intersection sequence (A, B, A), respectively (B, A, B).*

Proof. Let p, q be two points of the set $A+B$ for which the length of the segment pq attains a maximal value. Then neither p nor q can be an interior point of a trisecant of the set $A + B$, since if (for example) q were an interior point of a trisecant uv, then at least one of the segments pu, pv would be longer than pq, in contradiction to the definition of pq. □

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Lemma 5 cannot be strengthened as follows: on each of the two curves A, B there exists a point which is not the middle intersection point of a trisecant with intersection sequence (B, A, B), respectively (A, B, A). One can see this in the following example: Let A be a circle of radius r, given in polar coordinates by (1).

$$x = r \cdot \cos\phi, \quad y = r \cdot \sin\phi, \quad z = 0 \quad (0 \le \phi < 2\pi). \tag{1}$$

Let B be the curve

$$x = r \cdot \cos\phi \cdot \left(1 + \frac{1}{2}\cos\frac{\phi}{2}\right), \quad y = r \cdot \sin\phi \cdot \left(1 + \frac{1}{2}\cos\frac{\phi}{2}\right),$$
$$z = \frac{r}{2} \cdot \sin\frac{\phi}{2} \qquad (0 \le \phi < 4\pi). \tag{2}$$

B is described by the end point of a line of length $r/2$ lying in the normal plane of the circle A. If one leads the normal plane with constant velocity twice around A, and if one at the same time lets the line (with starting point A) rotate a full rotation in the normal plane with constant angular velocity against the main normal. Indeed, the two points of B belonging to the parameter values ϕ and $\phi + 2\pi$ determine a trisecant with intersection sequence (B, A, B) of which the middle intersection point is the point of A belonging to the parameter value ϕ. If the starting point of the trisecant runs through the curve once, the middle intersection point runs through the curve A twice. This example, which we have described for the sake of simplicity by continuous curves, can also be realized by polygons in general position.

The collection of all secants of the pair of polygons A, B with intersection sequence (A, B) is mapped continuously and injectively in a well understood manner to the product space of the two curves - a torus T. Let α and β be continuous, strictly monotone parameters of A, respectively B labeled in such a way that traversing A, respectively B once, corresponds to a change of the corresponding parameter value by ± 1. If one interprets $x = \alpha$ and $y = \beta$ as rectangular coordinates in a plane E, then E is the universal covering space of T with the deck transformations $(t_{m,n})\, x' = x + m$, $y' = y + n$ with integers m and n. Two points on E represent the same secant if and only if they are equivalent under a deck transformation.

A point x, y of the plane E is called a "representative" of a trisecant with intersection sequence (A, B, A) if the original points a, b on the polygons A, B belonging to the parameter values $x = \alpha$, $y = \beta$ are the first and middle intersection points of a trisecant with the intersection sequence (A, B, A). The point x, y is called an i-tuple representative of a trisecant with intersection sequence (A, B, A) if there are exactly i intersection points with A beyond b on the line ab oriented from a to b. Let $M(AB, A)$ be the subset of points of E consisting of the representatives p. 649
of a trisecant with intersection sequence (A, B, A). When counted, each point of $M(AB, A)$ has to be counted with the assigned multiplicity.

Using the corresponding convention about multiplicity, we will label $M(B, AB)$ as the set of representatives of trisecants with the intersection sequence (B, A, B); thereby each trisecant with this intersection sequence should be represented in the plane by such a system of equivalent points such that the original points a and b are the middle and end points of the trisecant with intersection sequence (B, A, B).

A point x, y belonging to the intersection of the two sets $M(AB, A)$ and $M(B, AB)$ represents the middle pair of intersection points of a quadrisecant with intersection sequence (B, A, B, A). If x, y is an i-tuple point of $M(AB, A)$ and a j-tuple point of $M(B, AB)$, then there are i-intersection points with A beyond b and j-intersection points with B before a on the line AB (which corresponds to the point x, y). Therefore there are $i \cdot j$ ways to interpret the line AB as a quadrisecant with intersection sequence (B, A, B, A) and middle pair of intersection points a, b. Therefore such a point x, y has to be counted $i \cdot j$ times as a point of the intersection $M(B, AB, A)$ of the sets $M(B, AB)$ and $M(AB, A)$. Conversely, each quadrisecant with intersection sequence (B, A, B, A) can only be obtained in one way from two trisecants with intersection sequence (B, A, B) and (A, B, A) by identifying the two last points of a trisecant with intersection sequence (B, A, B) with the two first points of a trisecant with intersection sequence (A, B, A). By the correct count of multiplicities, the relationship between quadrisecants with intersection sequence (B, A, B, A) and essentially different points (for example not equivalent under deck transformations) of the set $M(B, AB, A)$ is bijective. *Theorem 1 is therefore equivalent to the following claim: the set $M(B, AB, A)$ contains at least ${}_A\nu_B \cdot {}_B\nu_A$ essentially different points.*

The translation to the plane E of the results of Lemmas 4 and 5 is made possible by the following.

Lemma 6. *Each non-empty elementary part of the set of trisecants with intersection sequence (A, B, A) corresponds in the set $M(AB, A)$ to a system of equivalent closed double-point free arcs, which shall be called elementary arcs (Elementarbogen). Two equivalent elementary arcs are distinct from each other. The intersection* *of two essentially different elementary arcs, which are images of two elementary parts without a common vertex secant is either empty or a point of a (proper or not proper) sub-arc of the two arcs.*

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Proof. Depending on whether the elementary part $E_{\mathfrak{a}\mathfrak{b}\bar{\mathfrak{a}}}$ belongs to the system of generators of a one sheeted hyperboloid (or hyperbolic paraboloid) or a family of lines, the curve parameters are either associated to the secants of the elementary part one-to-one and continuously on A and on B OR one-to-one and continuously on A and single-valued and continuously on B. In each case $E_{\mathfrak{a}\mathfrak{b}\bar{\mathfrak{a}}}$ is mapped one-to-one and continuously in $M(AB, A)$. Hence, the image of $M(AB, A)$ is homeomorphic to a closed interval, that is to a closed double-point free arc. The vertex trisecants of $E_{\mathfrak{a}\mathfrak{b}\bar{\mathfrak{a}}}$ correspond to vertices of the elementary arc. - Since the curve parameters in $E_{\mathfrak{a}\mathfrak{b}\bar{\mathfrak{a}}}$ are restricted to the polygon edge, two equivalent elementary arcs have to be distinct from each other.

Two elementary arcs which correspond to two different elementary parts $E_{\mathfrak{a}_0\mathfrak{b}_0\bar{\mathfrak{a}}_0}$ and $E_{\mathfrak{a}_1\mathfrak{b}_1\bar{\mathfrak{a}}_1}$ can have a point in common only if $\mathfrak{a}_0 = \mathfrak{a}$ and $\mathfrak{b}_0 = \mathfrak{b}$.[15] The intersection of $E_{\mathfrak{a}_0\mathfrak{b}_0\bar{\mathfrak{a}}_0}$ and $E_{\mathfrak{a}_1\mathfrak{b}_1\bar{\mathfrak{a}}_1}$ corresponds in $M(AB, A)$ to a point or a simple arc, depending on whether the subinterval of $\bar{\mathfrak{a}}_1$ used to create $E_{\mathfrak{a}_1\mathfrak{b}_1\bar{\mathfrak{a}}_1}$ intersects the surface (formed by the generators of $E_{\mathfrak{a}_0\mathfrak{b}_0\bar{\mathfrak{a}}_0}$) in a point, or is totally or partially contained in this surface. - By the way, two non-equivalent elementary arcs of $M(AB, A)$ can also be distinct, if the corresponding elementary part $E_{\mathfrak{a}_0\mathfrak{b}_0\bar{\mathfrak{a}}_0}$ and $E_{\mathfrak{a}_1\mathfrak{b}_1\bar{\mathfrak{a}}_1}$ have a generator or a sub-interval of their family of generators in common. This can happen if $\mathfrak{a}_0 \neq \mathfrak{a}$, $\mathfrak{b}_0 = \mathfrak{b}_1$, $\bar{\mathfrak{a}}_0 = \bar{\mathfrak{a}}_1$. □

The composition of the set $M(AB, A)$ is completely described through the Corollary to Lemma 4 in connection with Lemma 6.

Lemma 7. *The set $M(AB, A)$ is composed of finitely many systems of equivalent elementary arcs. Lemma 6 determines the occurrence of multiple points belonging to the intersection of multiple essentially different elementary arcs. $M(AB, A)$ is the covering set of finitely many closed curves on the torus which are single-valued (but in general not one-to-one) continuous images of the component of the set of trisecants with intersection sequence (A, B, A).*

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The content of Lemma 5 is equivalent to

Lemma 8. *In E there exists either a line $y = const$ which is distinct from $M(AB, A)$ or a line $x = const$ which is distinct from $M(BA, B)$.*

The most important tool for the proof of Theorem 1 is formed by:

Lemma 9. *Each arc $\mathcal{C}$ in the plane E which connects a point x, y to a point (equivalent under a deck transformation $t_{m,1}$) $x + m$, $y + 1$ ($m = 0, \pm 1, \pm 2, \dots$) intersects the set $M(AB, A)$ in at least ${}_B\nu_A$ points.*

Proof. Without loss of generality one may assume that $\mathcal{C}$ connects the point $(0, 0)$ with the point $(m, 1)$. Let the number of intersection points of $\mathcal{C}$ with $M(AB, A)$

[15] The possibility that $\mathfrak{a}_0 = \mathfrak{a}$, $\mathfrak{b}_0 \neq \mathfrak{b}$ holds and that $\mathfrak{b}$ and $\mathfrak{b}_0$ have a vertex in common does not have to be considered, since in this case $E_{\mathfrak{a}_0\mathfrak{b}_0\bar{\mathfrak{a}}_0}$ and $E_{\mathfrak{a}_1\mathfrak{b}_1\bar{\mathfrak{a}}_1}$ have a vertex in common. In this case the two elementary arcs join each other to form a double-point free arc.

be finite - otherwise the claim is certainly true - and equal to r. Let a parametric representation of $\mathcal{C}$ be given by

$$(1) \qquad x = \phi(s) \quad y = \psi(s) \qquad (0 \leq s \leq 1);$$

where

$$(1') \qquad \phi(0) = 0, \quad \phi(1) = m,$$

$$(1'') \qquad \psi(0) = 0, \quad \psi(1) = 1.$$

If β is the parameter on B then

$$(2) \qquad \beta_1 = \psi(\beta) \qquad (0 \leq \beta \leq 1)$$

gives a single-valued continuous map of B onto itself. Indeed, ψ is a single-valued continuous function and condition $(1'')$ shows that continuity also holds in the point of B corresponding to the two parameter values $\beta = 0$ and $\beta = 1$. Furthermore, $(1'')$ implies that the degree of the map is $+1$. The map can therefore be derived from the identity through continuous deformations. Let B_1 be the image of B under the map (2).

The curve $\mathcal{C}$ associates to each point of B_1 a secant with intersection sequence (A, B) in the following way: to the point $b_1(s)$ of B_1 belonging to the parameter value $\beta_1 = \psi(s)$, a line is attached which intersects the polygon A in the point $a_1(s)$ belonging to the parameter value $\alpha_1 = \phi(s)$. The orientation of the line is from $a_1(s)$ to $b_1(s)$. The family of lines depends in a single-valued and continuous way on the parameter s. By $(1')$ and $(1'')$, $s = 0$ and $s = 1$ determine the same line. A line $a_1(s)b_1(s)$ has intersection points with A beyond $b_1(s)$ if and only if the corresponding point $\phi(s), \psi(s)$ of $\mathcal{C}$ belongs to $M(AB, A)$. Thereby the number of intersection points of $a_1(s)b_1(s)$ with A beyond $b_1(s)$ agrees exactly with the multiplicity of the point $\phi(s), \psi(s)$ in the set $M(AB, A)$. Hence a total number of exactly r such intersection points with A occur in the family of lines.

A separation of A and B with exactly r intersections of A and B can be achieved through the following - but not polygonal - deformation f_t $(0 \leq t \leq 2)$ of B: p. 652

(1) For $(0 \leq t \leq 1)$, let f_t be a deformation which transforms the identity map of B (to itself) into the map of B to itself given by equation (2), hence transforms B into B_1. No intersection of B with A occurs under this part of the deformation which only deforms the curve B in itself.

(2) For $(1 \leq t \leq 2)$, let $b_t(s)$ be the point lying beyond $b_1(s)$ at a distance of $\lambda(t-1)$ in the line $a_1(s)b_1(s)$. Therefore, during this part of the deformation, each point of the curve B_1 runs on the line associated to it with constant velocity λ in the direction of $a_1(s)b_1(s)$. Therefore only as many intersections of B with A can occur as there are intersection points with A beyond $b_1(s)$ in the family of lines, that is at most r. If one chooses the constant λ large enough so that the end result of the deformation, the curve B_2, lies completely outside of a sphere containing the polygon A in its interior, then the separation of A and B is attained.

By the way, in this case, the number of intersections of B with A is exactly equal to r.

The so defined deformation f_t $(0 \leq t \leq 2)$ can be approximated by a polygonal deformation f'_t, which also achieves the separation of A and B with at most r intersection of A and B. The definition of ${}_B\nu_A$ implies then ${}_B\nu_A \leq r$.

The creation of f_t' only depends on the second part $(1 \leq t \leq 2)$ of the deformation. Since if B_1' is any sufficiently good polygonal approximation of B_1 then $B_0' = B$ can be deformed polygonally in B_1' within a neighborhood of B which is distinct from A. Therefore for $(0 \leq t \leq 1)$ let f_t' be such a polygonal deformation which transforms B into the approximation B_1' defined as follows:- Let the number $\epsilon(> 0)$ be chosen in such a way that an ϵ-neighborhood of B is distinct from A and that an ϵ-neighborhood of B_2 remains outside of a sphere containing A in its interior. Let $b_t(s)$ be the image under the map f_t of the point of B belonging to the parameter value $\beta = s$. Let $s^{(\chi)}$ $(\chi = 1, 2, \ldots, q; q \leq r)$ be the parameter values for which intersection points of the segments $b_1^2(s^{(\chi)}) = b_1(s^{(\chi)})b_2(s^{(\chi)})$ with A occur. Each intersection point of $b_1^2(s^{(\chi)})$ with A has a positive minimal distance from those sides of A to which it does not belong. Let δ be the smallest among these distances. One determines $\sigma(> 0)$ in such a way that the segments $b_1^2(s)$ lie within a $\delta/2$ neighborhood of the $b_1^2(s^{(\chi)})$ for each interval $I^{(\chi)} : s^{(\chi)} - \sigma \leq s \leq s^{(\chi)} + \sigma$ $\chi = 1, 2, \ldots, q)$ such that furthermore the $I^{(\chi)}$ are distinct from each other for different χ and such that the parameter values belonging to the vertices of B lie outside of the $I^{(\chi)}$, except (possibly) for the $s^{(\chi)}$ and finally such that the sub-arcs of B_1 and B_2 corresponding to the interval $I^{(\chi)}$ can be approximated up to ϵ by the segment joining their endpoints. The distances of the points $b_t(s)$ from A, whose s-value does not belong to any of the $I^{(\chi)}$ remain above a positive bound η. One subdivides the complementary intervals of the $I^{(\chi)}$ finely enough such that for two consecutive partition points s' and s'' $(s' < s'')$ of the same complementary interval, all segments $b_1^2(s)$ with $s' \leq s \leq s''$ lie within a $\eta/2$ neighborhood of $b_1^2(s')$ and such that each sub-arc of B_1 and B_2 corresponding to an interval $s' \leq s \leq s''$ can be approximated up to ϵ by the segment joining its endpoints. Let $\bar{s}_v$ $(v = 1, 2, \ldots, n)$ be the s-values corresponding to a vertex of B, or which are $s^{(\chi)}$, $s^{(\chi)} - \sigma$, $s^{(\chi)} + \sigma$ or which are a partition points of on the complementary intervals of $I^{(\chi)}$. One orders the $\bar{s}_v$ corresponding to their ordering on B. If for any v the four points $b_1(\bar{s}_v)$, $b_1(\bar{s}_{v+1})$, $b_2(\bar{s}_v)$, $b_2(\bar{s}_{v+1})$, ($n+1$ has to be replaced by 1) lie in one plane, then one can achieve by an arbitrarily small perturbation of these points so that this is not the case anymore (that is not lying in a plane). This perturbation can be done in such a manner so that the points $b_1(s^{(\chi)})$ and $b_2(s^{(\chi)})$ remain fixed, so that the segments $b_1^2(\bar{s}_v)$ remain distinct from A for all $\bar{s}_v$ different from the $s^{(\chi)}$ and such that all previously made hypotheses about the approximation remain valid. The perturbation may transform the points $b_1(\bar{s}_v)$ and $b_2(\bar{s}_v)$ into the points $b_1(\bar{s}_v)'$, respectively $b_2(\bar{s}_v)'$. Let B_1' be the polygon formed by the segments joining $b_1'(\bar{s}_v)b_1'(\bar{s}_{v+1})$, B_2' be the polygon formed by the segments joining $b_2'(\bar{s}_v)b_2'(\bar{s}_{v+1})$ $(v = 1, 2 \ldots, n; n+1$ is replaced by 1). B_1' and B_2' approximate B_1 respectively B_2 up to ϵ. For $1 \leq t \leq 2$ let $b_t'(\bar{s}_v)$ be the point dividing the segment $b_1'(\bar{s}_v)b_2'(\bar{s}_v)$ in the ratio $(t-1) : (2-t)$. The deformation f_t' is defined for $1 \leq t \leq 2$ in the following way: each point of B which belongs to a parameter value $\beta = \bar{s}_v$ is mapped to the point $b_t'(\bar{s}_v)$ and each segment of B determined by two points $\beta = \bar{s}_v$ and $\beta = \bar{s}_{v+1}$ maps to the segment $b_t'(\bar{s}_v)b_t'(\bar{s}_{v+1})$. The construction of B_t' easily implies that the only intersections of B_t' with A are given by the intersection points of the segments $b_1^2(s^{(\chi)})$ with A. The edge lengths of the B_t' remain above a positive bound since

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the segments $b_1^2(\bar{s}_v)$ and $b_1^2(\bar{s}_{v+1})$, on which neighboring vertices move under the deformation, do not lie in one plane.[16] □

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Up to this point the set $M(AB, A)$ can be replaced by the set $M(B, AB)$ by interchanging B with A in all the lemmas. In particular, Lemma 9 will be used in the following for the set $M(AB, A)$ as well as for the set $M(B, AB)$. From now on we will distinguish one of the two sets because of Lemma 8: Let the set $M(AB, A)$ omit a family of equivalent lines $y = const$. Without loss of generality one may assume that these are the lines $y = 0, \pm 1, \pm 2, \ldots$.

Lemma 10. *The set $M(AB, A)$ (which omits the lines $y = 0, \pm 1, \ldots$ contains at least ${}_B\nu_A$ double-point free arcs $C_1, C_2, \ldots, C_{{}_B\nu_A}$, which satisfy the following conditions:*

(1) *$C_\nu(\nu = 1, 2, \ldots {}_B\nu_A)$ contains no pair of points equivalent under a deck transformation except for the endpoints.*
(2) *The endpoints of each C_ν correspond to each other under the deck transformation $t_{\pm 1,0}$.*
(3) *If i-arcs belonging to i-distinct C_ν (or curves equivalent to them) go through a point x, y of $M(AB, A)$, then x, y is (at least) an i-tuple point of the set $M(AB, A)$.*

Proof. If ${}_B\nu_A = 0$ then the lemma says nothing. Therefore for the proof assume ${}_B\nu_A > 0$. The lines $y = 0, \pm 1, \ldots$ are free of points of $M(AB, A)$. By Lemma 9 these lines are separated from each other by the set $M(AB, A)$ in the plane E. Let G be part of the component of the complement (Komplementärgebiet) of the set $M(AB, A)$ in E which contains the line $y = 0$. Let C_1^* be the part of the boundary of G which belongs to the strip $0 < y < 1$ and which separates $y = 0$ from $y = 1$. Since C_1^* is - like all of $M(AB, A)$ - built out of finitely many essentially different arcs and certain arcs equivalent to the essentially different arcs, C_1^* is a continuous curve, unbounded on both sides, which is mapped to itself under the deck transformation $t_{1,0}$ - as is G. Each point of C_1^* can therefore be reached from G. Any two points of the continuous curve C_1^* can be connected by a double-point free arc in C_1^*. Let C_1' be a double-point free arc which connects two points of C_1^* equivalent to each other under the deck transformation $t_{1,0}$. If C_1' contains another pair of points equivalent under $t_{1,0}$ besides the endpoints, then one can construct a subarc C_1 of C_1' on which the endpoints (but no pair of interior points) are equivalent under $t_{1,0}$.

If ${}_B\nu_A = 1$ then with the construction of C_1 the proof of the Lemma is complete. Therefore let us assume that ${}_B\nu_A > 1$. The simple open curve $\bar{C}_1 = \sum_{\mu=-\infty}^{+\infty} t_{\mu,0}(C_1)$ (which is unbounded on both sides) divides the plane between the two lines $y = 0$ and $y = 1$ in exactly two parts whose common border is $\bar{C}_1$. If one forms the set $M_1(AB, A) = M(AB, A) - \sum_{\mu,\nu=-\infty}^{+\infty} t_{\mu,\nu}(C_1)$, that is if one decreases by one the multiplicity of each point in $M(AB, A)$ which belongs to C_1 or to an arc which is equivalent to C_1, then the set $M_1(AB, A)$ is intersected in at

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[16]For the sake of the later application of the same method, let the following be added: if $\bar{s}_{v-1}$, $\bar{s}_v$, $\bar{s}_{v+1}$ are three consecutive partition points of the parameter interval and if the intersection points with A (which belong to the t-values $t^{(\chi)}$ $(\chi = 1, 2, \ldots, k)$)lie on the segment $b_1^2(\bar{s}_v)$, then by the perturbation of the vertices of $b_1^2(\bar{s}_{v-1})$ $b_1^2(\bar{s}_{v+1})$, one may also show that the two segments $b'_{t^{(\chi)}}(\bar{s}_{v-1})b'_{t^{(\chi)}}(\bar{s}_v)$ and $b'_{t^{(\chi)}}(\bar{s}_v)b'_{t^{(\chi)}}(\bar{s}_{v+1})$ are distinct for every $t^{(\chi)}$ up to the common vertex $b'_{t^{(\chi)}}(\bar{s}_v)$.

least ${}_B\nu_A - 1$ points by each arc $\mathcal{C}$ which connects a point of the line $y = 0$ with a point of the line $y = 1$. Indeed, let r be the number of intersection points of $\mathcal{C}$ with $M_1(AB, A)$. $\mathcal{C}$ intersects $\bar{C}_1$ at least once. If $\mathcal{C}$ and $\bar{C}_1$ possess more than one intersection point, then one replaces the sub-arc of $\mathcal{C}$ (from the starting point lying on $y = 0$ to the last intersection point q with $\bar{C}_1$) by a different arc which lies completely in the interior of G except for the endpoint q. The arc $\mathcal{C}_1$, thereby obtained from $\mathcal{C}$, has at most r-intersection points with $M_1(AB, A)$, exactly one intersection point with $\bar{C}_1$ and no more intersection points with the curves equivalent to $\bar{C}_1$ belonging to the half-plane $y < 0$. If $\mathcal{C}_1$ (corrected - mistake in Pannwitz??) still has intersection points with the curves equivalent to $\bar{C}_1$ lying in the half-plane $y > 1$, then one can replace the sub-arcs of $\mathcal{C}_1$ lying in $y \geq 1$ by segments of the line $y = 1$. The resulting arc $\mathcal{C}_2$ intersects $M_1(AB, A)$ in at most r points, the curve $\bar{C}_1$ in exactly one point and it does not intersect the curves equivalent to $\bar{C}_1$. Since one can connect the endpoint of $\mathcal{C}_2$ with a point equivalent to the start point by a segment of the line $y = 1$, without meeting $M(AB, A)$, the number of intersection points of $\mathcal{C}_2$ with $M(AB, A)$ must be by Lemma 9 at least equal to ${}_B\nu_A$, hence $r \geq_B \nu_A - 1$. Therefore $\mathcal{C}_2$ possesses at most $(r + 1)$-intersection points with $M(AB, A)$.

If one applies the method used for the section of C_1 to the set $M_1(AB, A)$, then one obtains a second arc C_2 of the sought type and a residue set $M_2(AB, A)$ which is intersected in at least ${}_B\nu_A - 2$ points by each arc $\mathcal{C}$ connecting two points of $y = 0$ and $y = 1$. If one already has constructed $C_1, C_2, \ldots, C_n$ using this method, then the residue set $M_n(AB, A)$ possesses as least $({}_B\nu_A - n)$-intersection points with each arc $\mathcal{C}$ connecting a point of $y = 0$ with a point of $y = 1$. If $n <_B \nu_A$, then the repetition of this method leads to a new curve C_{n+1}.

If follows immediately from the construction that the arcs $C_1, C_2, \ldots C_{{}_B\nu_A}$ constructed in this manner satisfy the conditions (1)-(3) of the lemma. □

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Proof. of Theorem 1: Each of the arcs $C_1, C_2, \ldots C_{{}_B\nu_A}$ constructed according to Lemma 10 has at least ${}_A\nu_B$ intersection points with $M(B, AB)$ by Lemma 9 applied to the set $M(B, AB)$. Two intersection points lying on the same C_ν are not equivalent by Lemma 10(1). Two intersection points with $M(B, AB)$ belonging to different C_ν are different in the sense of multiplicity because of property (3) in Lemma 10. The intersection $M(B, AB, A)$ of $M(AB, A)$ and $M(B, AB)$ therefore contains at least ${}_A\nu_B \cdot_B \nu_A$ points respecting the multiplicity which are essentially different. This fact is equivalent to the statement of Theorem 1. □

3. The Knot Invariant k. Proof of Proposition 5.

For the proof of Proposition 5 the same method can be used that Dehn used for the proof of the " Lemmas" about removing open double-lines.[17] Since in the present case the occurrence of boundary singularities requires a small modification of Dehn's method, the method shall be restated here with the required modifications.

Let K be a simple closed polygon in $\mathbb{R}^3$. Let E be a disk bounded by K. The assumptions formulated in the introduction concerning general position are assumed to hold for K and E. In addition to these conditions we additionally require the following: at most two triangles of E intersect in one segment, at most

[17]Dehn, l.c.

three triangles of E intersect in one point. The boundary singularities are simple, that is the intersection points of K with triangles of E are all distinct from each other. These conditions can be satisfied by an arbitrarily small perturbation of the vertices of E.

Let $\mathcal{E}$ be an abstract original of E: $\mathcal{E}$ is triangulated, and each triangle $\mathcal{D}$ of $\mathcal{E}$ has an affine map onto a non-degenerate triangle D of E. As a result of the prohibition of multiple singular lines, one can uniquely trace the path of a "double-line" on E and the path of both its originals in $\mathcal{E}$. Let s be a "double-segment" of E, that is the intersection of two triangles D_1 and D_2 of E, and let p be one of its vertices. p belongs to the boundary in at least one of the two triangles D_1, D_2. Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be the two original triangles of D_1 respectively, D_2 in $\mathcal{E}$. Let $\mathfrak{s}_1$ and $\mathfrak{s}_2$ be the two original segments of s in $\mathcal{D}_1$, $\mathcal{D}_2$. Since D_1, D_2 intersect in $\mathbb{R}^3$, $\mathcal{D}_1$ and $\mathcal{D}_2$ are either distinct from each other or they have exactly one vertex but no other point in common. First let $\mathcal{D}_1$, $\mathcal{D}_2$ be distinct from each other. If p belongs to p. 657
the boundary of D_1, then the original $\mathfrak{p}_1$ of p is an interior point of an edge $\mathfrak{d}_1$ of $\mathcal{D}_1$ and the original $\mathfrak{p}_2$ of p is an interior point of $\mathcal{D}_2$. Because of the general position of E, a vertex of D_1 cannot belong to D_2, nor can two edges of D_1 and D_2 intersect. If a triangle $\bar{\mathcal{D}}_1$ is incident (anschließt) to the edge $\mathfrak{d}_1$ of $\mathcal{D}_1$, then the image $\bar{D}_1$ of $\bar{\mathcal{D}}_1$ intersects the triangle D_2 in the segment $\bar{s}$, which connects to s in p. This continuation $\bar{s}$ of s is, together with its two original segments in $\mathcal{E}$, uniquely determined. If no triangle distinct from $\mathcal{D}_1$ connects to $\mathfrak{d}_1$ anymore, then $\mathfrak{d}_1$ belongs to the boundary $\mathcal{R}$ of $\mathcal{E}$, the image segment d_1 belongs to K and the point p is a boundary singularity of E. In this case one cannot continue the double-line beyond p; in E the originals of the double line $\mathfrak{p}_1$ and $\mathfrak{p}_2$ end and no originals of other double lines go through these two points because of the condition of simplicity of the boundary singularities. If $\mathcal{D}_1$ and $\mathcal{D}_2$ have a vertex $\mathfrak{e}_{12}$ in common, then the corresponding point e in $\mathbb{R}^3$ is an endpoint of the double segments. For the other endpoint - let it be called p - we are again in one of the two cases already considered. Therefore as an original of s, a segment $\mathfrak{s}_1 = \mathfrak{e}_{12}\mathfrak{p}_1$ lies in $\mathcal{D}_1$ and a segment $\mathfrak{s}_2 = \mathfrak{e}_{12}\mathfrak{p}_2$ lies in $\mathcal{D}_2$, and $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are mapped to s in such a way that $\mathfrak{e}_{12}$ maps to e, and $\mathfrak{p}_1$ and $\mathfrak{p}_2$ maps to p. In this case e is called a "winding point" of E. Further double lines could originate from e, resulting from other triangles with the common edge e intersecting in $\mathbb{R}^3$ in the same manner as D_1 and D_2. But, it is known that we can always avoid such "multiple winding points"[18], therefore we will assume in the following that all winding points of E are simple, that is end points of exactly one double line.

Altogether E consists of finitely many triangles, hence for the construction of double lines only finitely many double segments are available. If one follows a double line in one direction starting from an arbitrary double segment, then one of the following cases occurs:

(1) After finitely many steps one comes back to the starting segment - the double line is closed.
(2) The double line ends in a boundary singularity.

[18] As one easily sees, the method (applied to a double line originating from a multiple winding point) of "cutting" and "pasting" of $\mathcal{E}$ used for the removal of double lines in the following, leads to the splitting of the multiple winding point into several winding points with lower multiplicity. Thereby, the sum of the multiplicities of the new winding points is at most equal to the multiplicities of the original winding point, so that finitely many applications of this method leads to the goal.

(3) The double line ends in a winding point.

p. 658 In the last two cases the double line is open, therefore one of the cases (2) or (3) has to occur when following the same double line in the opposite direction. Hence we can have the following types of double lines on E: (a) closed double lines, (b) open double lines which have a boundary singularity at both endpoints, (c) open double lines which have a winding singularity at both endpoints, (d) open double lines which connect a boundary singularity with a winding point.

A uniquely determined double line of the type (b) or (d) originates from each boundary singularity of E. To each double line of type (b) is associated a pair a boundary singularities, namely its endpoints. Pairs of boundary singularities which belong to different double lines of type (b) are distinct from each other. Hence, if the number of boundary singularities of E is odd, at least one double line of type (d) occurs. In this case Dehn's method of "unshaltung" (switching?) along the open double lines leads to the removal of all double lines of type (d) without increase of the boundary singularities. After finishing the "switching" only an even number of boundary singularities can remain which is necessarily smaller than the original number of boundary singularities of E. The necessary number of boundary singularities for a disk bounded by K therefore has to be even since it cannot be decreased anymore.

First we will describe the "*removal of the double line of type* (d)" in an especially simple case. Let l be a double line of type (d) which connects a boundary singularity p with a winding point e. Assume that the originals $\mathfrak{l}_1$, $\mathfrak{l}_2$ of l are double-point free and distinct except for $\mathfrak{e}_{12}$, the original of the winding point of e. One of the originals of p - let it be the endpoint $\mathfrak{p}_1$ of $\mathfrak{l}_1$ - lies on the boundary $\mathcal{R}$ of E. The other - the endpoint $\mathfrak{p}_2$ of $\mathfrak{l}_2$ - lies in the interior of a triangle of $\mathcal{E}$. The labels $\mathfrak{l}_1$, $\mathfrak{l}_2$ shall be used for the lines with the orientation from $\mathfrak{e}_{12}$ to $\mathfrak{p}_1$ or $\mathfrak{p}_2$. The lines oriented in the opposite direction are called, as usual, $\mathfrak{l}_1^{-1}$, $\mathfrak{l}_2^{-1}$. The triangulation of $\mathcal{E}$ is refined so that the lines $\mathfrak{l}_1$ and $\mathfrak{l}_2$ lie on edges of the refined triangulation and so that points of $\mathfrak{l}_1$ and $\mathfrak{l}_2$ which are mapped to the same point of l are simultaneously (either) vertices or interior points of edges of the triangulation. Then one cuts the disk $\mathcal{E}$ along the double-point free line $\mathfrak{l}_1^{-1}\mathfrak{l}_2$, but without cutting the boundary of $\mathcal{E}$ in the point $\mathfrak{p}_1$. From that results a hole in $\mathcal{E}$ touching the boundary $\mathcal{R}$. The hole is bounded by $\bar{\mathfrak{l}}_1^{-1}\bar{\mathfrak{l}}_2\bar{\bar{\mathfrak{l}}}_2^{-1}\bar{\bar{\mathfrak{l}}}_1$, if one distinguishes the two sides of the line $\mathfrak{l}_1^{-1}\mathfrak{l}_2$ by using
p. 659 "bars". If one now glues $\bar{\mathfrak{l}}_1$ with $\bar{\mathfrak{l}}_2$, $\bar{\bar{\mathfrak{l}}}_1$, with $\bar{\bar{\mathfrak{l}}}_2$ in the way described by the map onto l, then a surface $\mathcal{E}'$ results, which is obviously again a disk. Since $\mathcal{E}'$ consists of the same triangles as $\mathcal{E}$ (and furthermore the points which correspond the the same image under the map from $\mathcal{E}$ to E, get identified with each other under the gluing of the boundary of the hole,) the map from $\mathcal{E}$ to E gives a map from $\mathcal{E}'$ to a disk E' of $\mathbb{R}^3$. E' is identical to E as a point set, but is different from E, triangulated according to the refined triangulation of $\mathcal{E}$, by the way the triangles along l are glued. Under the map of $\mathcal{E}'$ to E', the point p has only a single original point $\mathfrak{p}_{12}$ lying on the boundary $\mathcal{R}$ of $\mathcal{E}'$ since under the identification $\mathfrak{p}_1$ and $\mathfrak{p}_2$ were glued to each other. Hence p is not a boundary singularity of $\mathcal{E}'$ anymore. The point e has two originals $\bar{e}_{12}$ and $\bar{\bar{e}}_{12}$ in $\mathcal{E}'$ which resulted from $\mathfrak{e}_{12}$ under the cutting of $\mathcal{E}$ and which were not identified under the gluing. l has two originals $\bar{l}_{12}$ and $\bar{\bar{\mathfrak{l}}}_{12}$ which resulted from the identification of $\bar{\mathfrak{l}}_1$ with $\bar{\mathfrak{l}}_2$, respectively of $\bar{\bar{\mathfrak{l}}}_1$ with $\bar{\bar{\mathfrak{l}}}_2$ and which connect $\bar{e}_{12}$, respectively $\bar{\bar{e}}_{12}$ with $\mathfrak{p}_{12}$. If s is a subsegment of l and if $\bar{\mathcal{D}}_1$, $\bar{\bar{\mathcal{D}}}_1$, $\bar{\mathcal{D}}_2$, $\bar{\bar{\mathcal{D}}}_2$ are those triangles of the refined triangulation of $\mathcal{E}$ which

border/end on the originals $\bar{s}_1$, $\bar{\bar{s}}_1$, $\bar{s}_2$, $\bar{\bar{s}}_2$ of s in the "cut open" disk, then while traversing s cyclically the images $\bar{\mathcal{D}}_1$, $\bar{\bar{\mathcal{D}}}_1$, $\bar{\mathcal{D}}_2$, $\bar{\bar{\mathcal{D}}}_2$ in $\mathbb{R}^3$ follow each other in the ordering $\bar{\mathcal{D}}_1$, $\bar{\mathcal{D}}_2$, $\bar{\bar{\mathcal{D}}}_1$, $\bar{\bar{\mathcal{D}}}_2$ (or in the opposite ordering). (why?) Since $\bar{\mathcal{D}}_1$ and $\bar{\bar{\mathcal{D}}}_1$ are sub-triangles of a triangle D_1 of the original triangulation of E and $\bar{\mathcal{D}}_2$ and $\bar{\bar{\mathcal{D}}}_2$ are sub-triangles of a triangle D_2 as well, and D_1 and D_2 intersect in s. Along s $\bar{\mathcal{D}}_1$ and $\bar{\mathcal{D}}_2$, $\bar{\bar{\mathcal{D}}}_1$ and $\bar{\bar{\mathcal{D}}}_2$ are glued together in E', therefore the double line l represents a "self-tangency" (selbstberührung) in E'. Through a suitable arbitrary small perturbation of the vertices of E', one can restore the general position of E' (with the additional conditions mentioned at the beginning of this section) and the self-tangency of E' can be removed without the occurrence of new singularities.

If E is any disk bounded by K which possesses an odd number of boundary singularities, and hence at least one double line of type (d), then the described reduction of the number of boundary singularities can in general not be done, since the hypotheses made for the removal of the double lines need not be satisfied. This case is modified to the previous case by the following *reductions*: Let e be a winding point of E. Let l be an open double line of type (d) originating from e whose originals $\mathfrak{l}_1$, $\mathfrak{l}_2$ which originate from the point $\mathfrak{e}_{12}$ have double points or intersect in $\mathcal{E}$. If q is an intersection point of $\mathfrak{l}_1$ and $\mathfrak{l}_2$, then the point $q = q_1$ of p. 660
$\mathfrak{l}_1$ corresponds to a point q_2 different from q_1 via the correspondence given by l. (Why?) Otherwise q would be a winding point and therefore a common endpoint of the lines $\mathfrak{l}_1$ and $\mathfrak{l}_2$. Because of this, one can determine a point p on l such that (1) the originals $l_1^* = \mathfrak{e}_{12}\mathfrak{p}_1$ and $l_2^* = \mathfrak{e}_{12}\mathfrak{p}_2$ of the part l^* of l determined by e and p are distinct from each other and double-point free; (2) and such that there is a double point of $\mathfrak{l}_1$ or $\mathfrak{l}_2$ or an intersection point of $\mathfrak{l}_1$ with $\mathfrak{l}_2$ on at least one of the lines $\mathfrak{l}_1^*$, $\mathfrak{l}_2^*$. One can proceed similarly to the previous case. The triangulation of E is refined in such a way that $\mathfrak{l}_1^*$, $\mathfrak{l}_2^*$ are edges of the triangulation and corresponding points of $\mathfrak{l}_1^*$ and $\mathfrak{l}_2^*$ are simultaneously vertices or interior points of edges of the triangulation. Let $\mathfrak{l}_1^*$ and $\mathfrak{l}_2^*$ be the lines oriented from $\mathfrak{e}_{12}$ to $\mathfrak{p}_1$ and $\mathfrak{p}_2$. One cuts the disk E along the double-point free line $\mathfrak{l}_1^{*-1}\mathfrak{l}_2^*$. This gives a hole in $\mathcal{E}$ bounded by $\bar{\mathfrak{l}}_1^{*-1}\bar{\mathfrak{l}}_2^*\bar{\bar{\mathfrak{l}}}_2^{*-1}\bar{\bar{\mathfrak{l}}}_1^*$ (these "bars" again distinguishing the two sides of the line $\mathfrak{l}_1^{*-1}\mathfrak{l}_2^*$). If one now glues $\bar{\mathfrak{l}}_1^*$ with $\bar{\mathfrak{l}}_2^*$ and $\bar{\bar{\mathfrak{l}}}_1^*$ with $\bar{\bar{\mathfrak{l}}}_2^*$, such that the points which coincide under the map l are identified with each other, then one again obtains a new disk $\mathcal{E}'$. The map from $\mathcal{E}$ to E also defines a map from $\mathcal{E}'$ to a disk E' which is identical to E as a point set. The point e has two originals $\bar{e}_{12}$ and $\bar{\bar{e}}_{12}$ in $\mathcal{E}'$ - these were created from $\mathfrak{e}_{12}$ under the cutting. In contrast, the point p has only one original $\mathfrak{p}_{12}$ in $\mathcal{E}'$ resulting from the identification of $\mathfrak{p}_1$ and $\mathfrak{p}_2$. Originals of l in $\mathcal{E}'$ are two distinct (except for $\mathfrak{p}_{12}$) double-point free lines $\bar{l}_{12}$ and $\bar{\bar{\mathfrak{l}}}_{12}$ which resulted from the identification of $\bar{\mathfrak{l}}_1$ with $\bar{\mathfrak{l}}_2$ and respectively of $\bar{\bar{\mathfrak{l}}}_1$ with $\bar{\bar{\mathfrak{l}}}_2$ and which connect $\bar{e}_{12}$ respectively $\bar{\bar{e}}_{12}$ with $\mathfrak{p}_{12}$. The line l is - as before - a self-tangency of the disk $\mathcal{E}'$, which can be removed by an arbitrarily small perturbation of the vertices of $\mathcal{E}'$ without the occurrence of new singularities. Simultaneously, one can (by this perturbation) re-obtain the general position of $\mathcal{E}'$ with the additional conditions made in the beginning of this section. Under the perturbation, the point e is split into two points corresponding to the existence of two originals in $\mathcal{E}'$. In contrast, the point p remains unsplit. A double line originates from p which in the beginning coincides with the part of the old double line l not belonging to l^*. Indeed, the originals of the part of l not belonging to l^* both start in the point $\mathfrak{p}_{12}$ in $\mathcal{E}'$ because of the identification of $\mathfrak{p}_1$ and $\mathfrak{p}_2$. Therefore p became a winding point. In its further part, the double line p. 661

originating from p will in general not coincide with the old double line l. (why?) Since under the cut of $\mathcal{E}$ along $\mathfrak{l}_1^{*-1}\mathfrak{l}_2^*$ the originals of l were cut in each double point lying on $\mathfrak{l}_1^{*-1}\mathfrak{l}_2^*$ OR in each intersection point of $\mathfrak{l}_1$ and $\mathfrak{l}_2$ and were glued back together under the identification in a difference way or they were glued with other (under the cut) cut originals of double lines. In particular, it is not necessary that the double line originating from p in E' ends in the same point as the old double line l.

After the "reduction" the number of boundary singularities of E' is the same as those of E, hence also odd. Hence, there is again at least one double line l' of type (d). Either this double line - and therefore the boundary singularity in which it ends - can be removed by the first described method, OR the originals of l' in $\mathcal{E}'$ have double points or intersection points with each other, such that the "reduction" can be applied again to $\mathcal{E}'$ using the double line l'. Each time, the reduction leads to a decrease of the number z of double points occurring in the original set of all double lines. Since by hypothesis, at least one of the originals $\mathfrak{l}_1^*$ and $\mathfrak{l}_2^*$ of the line l^* (removed by reduction) possesses at least one intersection point of the original of the part of l not belonging to l^*, and this intersection point vanishes under the removal of l^*.[19] Since z is finite, this reduction can be repeated only a finite number of times, therefore at least once - and namely after at most z-reductions - a case has to occur in which the removal of a double line of type (d) (and necessarily of the boundary singularity belonging to it,) is possible.

4. Multiple Secants of a Knot - Proof of Theorem 2.

The proof of Theorem 2 can be done using the same method as the case of Theorem 1. Each quadrisecant of the simple closed polygon K is regarded as a line on which the first and third intersection points of a trisecant coincide with the first
p. 662 and second intersection points of another trisecant.

In the following, let K be a simple closed polygon in $\mathbb{R}^3$ in general position and let k be the knotting number of K.

Lemma 11. [20] *Each point of K is a starting point for at least k trisecants of K.*

Proof. Let p_0 be a fixed point of K. Let r be the number of trisecants originating from p_0. Let K' be the connected set of subsegments (Streckenzug) of K consisting of all and only the segments of K which p_0 does not belong to. If the point p runs through K' then the secants p_0p span a disk E bounded by K. Each time that there is another intersection point q of p_0p with K between p_0 and p, q is a boundary singularity of E. Additional boundary singularities can not occur, in particular p_0 is also not a boundary singularity of E, hence the number of boundary singularities of E is equal to r, and thereby $r \geq k$. □

Within the set of trisecants of K we can define neighborhood for the set of secants with intersection sequence (A, B, A) in the same way as we did in Section 2. An elementary part E_{123} of the set of trisecants will again be the collection of all

[19]At this point the hypotheses of the simplicity of the winding point e is used. If e were a winding point of sufficiently high multiplicity, then under the cutting and gluing of $\mathcal{E}$ multiple pairs of the double lines in E' ending in e could each form a double line and therefore new double points could occur in the system of originals of double lines of E', which would result in an increase of z.

[20]Compare footnote 6.

trisecants of K, whose first, second and third intersection points with K belong to three given sides s_1, s_2 and s_3 of K in that ordering.[21] The study of the elementary parts of the set of trisecants is done in the same way as in Section 2. In addition to the possible structure for an elementary part described there another now occurs. An elementary part of the set of trisecants of K can be homeomorphic to an interval (closed at one end.) Because, if the two edges s_1 and s_2 have a vertex in common and s_3 intersects the plane determined by s_2 and s_2 in a point p_3 such that the elementary part E_{123} is non empty, then the trisecants of E_{123} cluster in a line ep_3 corresponding to an end point of an interval, but no longer belonging to E_{123} since the line possesses only two intersection points with K. The analogue to Lemma 6 is therefore:

Lemma 12. *Each nonempty elementary part of the set of all trisecants of K is homeomorphic to either an interval closed on one end or to a closed interval. Each non-empty elementary part contains - depending on which of the two cases occurs - exactly one or exactly two vertex trisecants which correspond to one or both end points of the interval belonging to the interval. The intersection of two elementary parts is either empty or consists of exactly one vertex trisecant. Each vertex trisecant belongs to exactly two elementary parts.*

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From Lemma 12 results for the structure of the set of trisecants of K similar to section 2.

Corollary to Lemma 12. *The set of trisecants of K consists of finitely many components. Each component is either homeomorphic to a simple closed curve or a simple open arc.*

We define a map of the secants p_1p_2 of K to the universal covering space of an annulus, a strip S bounded by two parallel lines, by the following rule: Let χ be a continuous and essentially monotone parameter on the polygon K labeled in such a way that its value changes by ± 1 after traversing K once. Let χ_1 and χ_2 be the parameter values corresponding to p_1 respectively p_2 which are determined mod 1. Let the image points of the secant p_1p_2 in the strip S: $0 \leq y \leq 1$ of the xy-plane be those points whose coordinates are given by

$$x = \chi_1, \quad y \equiv \chi_2 - \chi_1 \pmod 1, \qquad 0 \leq \mathrm{y} \leq 1.$$

If p_1 and p_2 are distinct, hence if $\chi_2 - \chi_1$ is not an integer, this associates to each oriented secant of K exactly one system of points, which are equivalent to each other under the deck transformations of the strip $(t_m) : x' = x + m$, $y' = y$ $(m = 0, \pm 1, \pm 2, \dots)$. Different secants correspond to different systems of points. If $\chi_2 - \chi_1$ is an integer, hence if $p_1 = p_2$ and the secant has degenerated to a point, then two different systems of equivalent points correspond to it, namely one system on the line $y = 0$ and one system on the line $y = 1$.

To the set of trisecants of K, we associate two images (defined in different ways) in the strip S.

Let M_{12} be the set of points of S which are an image of the sub-secant p_1p_2 of a trisecant $p_1p_2p_3$ of K. A point x, y of S has to be counted i-times as a point of the

[21] While we disregarded the orientation of the secants when formulating Theorem 2 in the introduction, it is useful for the proof to again regard the secants as being oriented.

set M_{12} if on the trisecant associated to it there are i further intersection points
p. 664 with K beyond p_2.[22]

Let M_{13} be the set of those points of S which are an image of a secant p_1p_3 determined by a trisecant $p_1p_2p_3$. A point x, y has to be counted j-times as a point of the set M_{13} if on the trisecant p_1p_3 associated to it, there are j further intersection point with K between p_1 and p_3.[22]

If a point x, y belongs to the intersection M_{123} of M_{12} and M_{13}, then its original p_1p_3 represents a first and third intersection point of a quadrisecant of K. If x, y is an i-tuple point of M_{12} and a j-tuple point of M_{13}, then there are $i \cdot j$ possible ways to interpret p_1 and p_3 as the first and third point of a quadrisecant of K. Accordingly such a point has to be counted $(i \cdot j)$ times as a point of M_{123}. Conversely, each quadrisecant $p_1p_2p_3p_4$ of K (respecting its orientation) is represented only by a system of equivalent points in M_{123} corresponding to its first and third intersection point. But the two secants $p_1p_2p_3p_4$ and $p_4p_3p_2p_1$ (which are to be regarded as not different for Theorem 2) have essentially different image points in M_{123}. Therefore for the proof of Theorem 2 it remains to show: The intersection M_{123} of M_{12} and M_{13} (under the correct count of multiplicity) contains at least k^2 essentially different points.

The mapping of the elementary parts of the set of trisecants of K is exactly done by Lemma 6, with the single difference that the elementary parts open on one side correspond to elementary arcs under exclusion of one end point. Accordingly, the following holds:

Lemma 13. *Each of the sets M_{12} and M_{13} consist of finitely many essentially different simple elementary arcs, possibly excluding one end point, and the arcs equivalent to them. The occurrence of multiple points which belong to the intersection of multiple essentially different elementary arcs without a common endpoint is described by Lemma 7.*

Lemma 14. *The set M_{13} has a positive minimal distance from the lines $y = 0$ and $y = 1$.*

Proof: From the study of the elementary parts of the set of trisecants it follows easily that the first and third intersection points of a trisecant cannot converge to the same point of K when approaching the boundary of an elementary part.

Lemma 15. *Each arc $\mathcal{C}$ in the interior of S (therefore in $0 < y < 1$), which connects a point x, y with the point $x + 1, y$ equivalent to it under t_1, intersects the set M_{12} in at least k-points.*

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Proof. Without loss of generality, one may assume that $\mathcal{C}$ connects a point with the coordinates $0, y_0$ with the point $1, y_0$. Let $\mathcal{C}$ have r-intersection points with M_{12}.

Let A be a parametric representation of $\mathcal{C}$ be given by

$$x = \phi(s), \quad y = \psi(s) \quad (0 \le s \le 1), \tag{1}$$

whereby we have

$$\phi(0) = 0,\ \phi(1) = 1, \qquad \psi(0) = y_0,\ \psi(1) = y_0. \tag{1'}$$

[22]Because of the general position, the condition that two intersection points of a trisecant of K never belong to the same polygon edge is automatically satisfied. Consequently, the multiplicity of the points of M_{12} and M_{13} is always finite.

Through

$$(2) \qquad \chi_2 = \phi(\chi) + \psi(\chi) \qquad (0 \le \chi \le 1),$$

a map from K onto itself of degree $+1$ is defined because of the condition $(2')$ following from (1).

$$(2') \qquad \chi_2(1) = \chi_2(0) + 1.$$

The image K_2 of K under $\chi_2(\chi)$ can be obtained through a continuous deformation of K (to itself).

To each point $p_2(\chi)$ corresponding to the parameter value $\chi_2(\chi)$ a secant $p_1(\chi)p_2(\chi)$ ending in $p_2(\chi)$ of K is associated using the curve $\mathcal{C}$. This is done by the following rule: Let $p_1(\chi)$ be the point of K belonging to the parameter value $\phi(\chi)$. This family of secants depends continuously on χ and closes because of $(1')$ after traversing K_2 once. Such a secant $p_1(\chi)p_2(\chi)$ possesses an intersection point with K beyond $p_2(\chi)$ only if the point of $\mathcal{C}$ belonging to the parameter value $s = \chi$ belongs to M_{12}. If this point is an i-tuple point of M_{12}, then there are i-intersection points with K beyond $p_2(\chi)$ on the secant. Therefore there are altogether r-intersection points with K beyond the point $p_2(\chi)$ in the family of secants.

First we construct a disk E_2 bounded by K_2 - but not consisting of triangles - which is intersected by K in exactly r interior points. In order to do this one attaches to each point $p_2(\chi)$ of K_2 the part of the half-line $p_1(\chi)p_2(\chi)$ beyond $p_2(\chi)$. All these rays are cut on a sufficiently large sphere Q containing K in its interior and into the intersection curve K_3 on Q we span a disk lying on Q or in the exterior of Q. Intersection points of the disk E_2 defined in this way with K are obviously all points and only the points in which the secants $p_1(\chi)p_2(\chi)$ intersect the polygon K beyond $p_2(\chi)$. By changing the disk E_2 in the vicinity of the boundary K_2, one can replace E_2 by a disk E bounded by K that possesses exactly r boundary singularities. The following construction leads to this result: let K_2^* be the parallel curve of K_2 lying on E_2 whose points $p_2^*(\chi)$ $(0 \le \chi \le 1)$ lie beyond $p_2(\chi)$ at a fixed distance $\lambda(>0)$ on the rays $p_1(\chi)p_2(\chi)$. Here $\lambda(>0)$ is p. 666
chosen such that the part of E_2 between K_2 and K_2^* contains no intersection point with K. Therefore the sub-disk E_2^* of E_2 bounded by K_2^* is intersected exactly r-times by K. Now one encloses the polygon K in a torus T of such a small diameter that K_2^* lies completely outside of T. On T there exists a double-point free curve K' which is homotopic to K_2^* in the exterior of T and which therefore bounds (there) together with K_2^* a strip (with two edges) B^*. B^* is in general not singularity free. K' is homotopic to K_2 in the interior of T therefore it is also homotopic to K and K' and K obviously bound even a singularity free double edge strip B in the interior of T. One obtains a disk E bounded by K with exactly r boundary singularities if one attaches to the boundary K_2^* of E_2^* the strip B^* and then to the free boundary K^* of B^* the strip B.

If one replaces the disk E (which is not yet triangulated therefore is not considered in the definition of k), by an approximation satisfying the conditions from the introduction and having at most r boundary singularities, then the definition of k implies that $r \ge k$.

Under the process of the approximation of E it obviously only depends on the approximation of the strip bounded by K_2^* and K_3, which contains the boundary singularities of E. Because the torus T , the curve K' and the strip B can from the beginning be constructed out of triangles and straight segments respecting the condition of general position. Furthermore B^* and the part of E lying outside of

the sphere Q - both of which are distinct from K - can be approximated so well that the approximation is also distinct from K. In order to recognize the possibility of an approximation of the surface F bounded by K_2^* and K_3 without increasing the intersection points with K, one has to show that the intersection points of K with F are simple. That is that a sufficiently small neighborhood of such an intersection point on F can be interpreted as a one-to-one continuous image of a disk. Since an intersection point with K not satisfying the condition of simplicity is in general split into multiple intersection points under the approximation of F. Instead of proving the simplicity of the intersection points with K for F itself, one can replace F with a surface F' which has exactly as many intersection points with K as F and which approximates F with the required precision. Such a surface F' - by the way not yet triangulated - is immediately given through the construction of a polygonal deformation of B_1' into B_2' which was used in the proof of Lemma
p. 667 9 (last paragraph). Because if one has defined a polygonal approximation K_2' of K_2^*, a polygonal approximation K_3' of K_3 and a polygonal deformation of K_2' into K_3' respecting the additional conditions in footnote 16 (with K now playing the role of the polygon A in Lemma 9), then the required simplicity of the intersection points is obviously satisfied for the surface F' swept out (überstrichene) by the deformation of K_2'. The approximation of F' though a triangulated surface causes no additional difficulties and does not need to be done here. □

Lemma 16. *Each arc $\mathcal{C}$ in the strip S which connects a point of the line $y = 0$ with a point of the line $y = 1$ intersects the set M_{13} in at least k points.*

Proof. Abutting (anschließen) the boundaries $y = 0$ and $y = 1$ of S are the two (closed) areas S_0 respectively S_1 which correspond to the secants of K which lie completely in one side of K. If p_0 is a fixed point of K, $x = x_0, 0 \leq y \leq 1$ a segment S whose x-value corresponds to the point p_0, then to the secants beginning in the point p_0 whose endpoints in K lie in the direction of increasing parameter values between p_0 and the next vertex following after p_0, belong such points of S which lie on a segment $x = x_0, 0 \leq y \leq \bar{\eta}_0\,(\bar{\eta}_0 > 0)$. Thereby $\bar{\eta}_0$ denotes the difference in the parameter between p_0 and the vertex of K following p_0. The segment $x = x_0, 0 \leq y \leq \eta_0$ forms the intersection of the segment $x = x_0, 0 \leq y \leq 1$ with S_0. Correspondingly, the intersection of S_1 with $x = x_0, 0 \leq y \leq 1$ is a segment $x = x_0, 0 \leq y \leq 1 - \bar{\bar{\eta}}_0\,(\bar{\bar{\eta}}_0 > 0)$, which corresponds to those secants starting in p_0 whose endpoints lie between p_0 and the first vertex of K preceding p_0. The sets S_0 and S_1 still have a positive minimal distance from M_{13}.

We are allowed to arbitrarily change the curve $\mathcal{C}$ in the vicinity of the lines $y = 0$ and $y = 1$ since this does not change the number of its intersection points with M_{13}. One may therefore make the following assumptions: $\mathcal{C}$ connects the point $(0, 0)$ with the point $(0, 1)$; the beginning of $\mathcal{C}$ coincides with the intersection of the segment $x = 0, 0 \leq y \leq 1$ with S_0; the end of $\mathcal{C}$ coincides with the intersection of the segment $x = 0, 0 \leq y \leq 1$ with S_1; except for these start and end segments, $\mathcal{C}$ is distinct from S_0 and S_1. The number of intersection points of $\mathcal{C}$ with M_{13} is equal to r.

Let a parametric representation of $\mathcal{C}$ be given by

$$(1) \qquad x = \phi(s), \quad y = \psi(s) \qquad (0 \leq s \leq 1),$$

where we have

$$(1') \qquad \phi(0) = 0, \quad \phi(1) = 0,$$

$$(1'') \qquad \psi(0) = 0, \quad \psi(1) = 1,$$

Because of $(1')$ a map from K to itself of degree zero is given by (2) p. 668

$$(2) \qquad \chi_1 = \phi(\chi) \qquad (0 \leq \chi \leq 1).$$

The image K_1 can (consequently) be continuously contracted to a point. A second map of K to itself is defined by (3)

$$(3) \qquad \chi_3 = \phi(\chi) + \psi(\chi) \quad (0 \leq \chi \leq 1).$$

This map has degree $+1$ because of the equality $(3')$ following from $(1')$ and $(1'')$

$$(3') \qquad \phi(0) + \psi(0) = \phi(1) + \psi(1) - 1.$$

The image of K_3 can therefore be obtained by a continuous deformation of K in itself.

The points of the curve $\mathcal{C}$ correspond in $\mathbb{R}^3$ to exactly those secants of K, which connect a point $p_1(\chi)$ of K_1 with the point $p_3(\chi)$ of K_3 belonging to the same parameter value χ. The starting point $(0,0)$ of $\mathcal{C}$ corresponds to a secant degenerate to one point $p_1(0) = p_3(0)$. Then corresponding to the beginning part of $\mathcal{C}$ belonging to S_0 there are "following" secants whose start point coincides with $p_1(0)$ and whose end points lie between $p_1(0)$ and the next vertex $\bar{e}$ of K following $p_1(0)$. These secants are not all degenerate and they have - because of the general position of K - no subsegment in common with an edge of K, except for those secants which correspond to the end part of $\mathcal{C}$ belonging to S_1. The starting points of these last secants is again the point $p_1(0) = p_1(1)$. Their endpoints lie between $p_1(1)$ and the last vertex $\bar{\bar{e}}$ of K preceding the point $p_1(1)$. The end point of $\mathcal{C}$ corresponds again to the secant degenerate in $p_1(1) = p_3(1)$. Those secants which correspond to the part of $\mathcal{C}$ outside of S_0 and S_1 have exactly as many intersection points with K between $p_1(\chi)$ and $p_3(\chi)$ as the curve $\mathcal{C}$ possesses intersection points with M_{13}, therefore a total of r. The surface which is swept out by the segments $p_1(\chi)p_3(\chi)$ $(0 \leq \chi \leq 1)$ associated to the points of $\mathcal{C}$ can now - similarly as in the proof of the preceding lemma - be used for the construction of a disk with at most r boundary singularities and which is bounded by K. This again implies that $r \geq k$.

Let $\bar{\chi}$ be the smallest value of $\chi\,(0 \leq \chi \leq 1)$ for which $p_3(\bar{\chi}) = \bar{e}$ holds. Let $\bar{\bar{\chi}}$ be the largest value of $\chi\,(0 \leq \chi \leq 1)$ for which $p_3(\bar{\bar{\chi}}) = \bar{\bar{e}}$ holds. The subintervals $0 \leq \chi \leq \bar{\chi}$ and $\bar{\bar{\chi}} \leq \chi \leq 1$ of K therefore correspond - according to the special hypothesis about the position of $\mathcal{C}$ - to exactly those secants $p_1(\chi)p_3(\chi)$ which correspond to the start and end parts of $\mathcal{C}$ belonging to S_0 respectively S_1 and which lie in the segment $p_1(0)\bar{e}$ respectively $p_1(0)\bar{\bar{e}}$. Let $\bar{K}_3$ be the image p. 669 under map (3) of the part of K determined by $\bar{\chi} \leq \chi \leq \bar{\bar{\chi}}$. Furthermore, let $\bar{\bar{K}}$ be the part of K which is bounded by $\bar{e}$ and $\bar{\bar{e}}$ and which does not contain the point $p_3(0)$. Then $\bar{K}_3$ can be continuously deformed within K into $\bar{\bar{K}}$ while fixing the endpoints; this follows immediately form the fact that the image K_3 of K, which can be continuously deformed into K, consists of $\bar{K}_3$ and the two (simply covered under the map) segments $\bar{e}p_3(0)$ and $\bar{p}_3(0)\bar{\bar{e}}$, while K consists of $\bar{\bar{K}}$ and the same two segments. - Let $\bar{K}_3''$ be the parallel curve of $\bar{K}_3$ whose points $\bar{p}_3''(\chi)$ $\bar{\chi} \leq \chi \leq \bar{\bar{\chi}}$, lie beyond $p_3(\chi)$ on $p_1(\chi)p_3(\chi)$ at a fixed distance λ. Choose λ in such a way that there are no intersection points with K on the segments $\bar{p}_3''(\chi)p_3(\chi)$, furthermore λ should at most be equal to one third of the length of the segments $p_3(0)\bar{e}$ and $p_3(0)\bar{\bar{e}}$. - Let Z be the surface of a "cylindrical neighborhood" of $\bar{\bar{K}}$, that is the surface of a one-to-one continuous image of a cylinder (bounded on both sides) which satisfies the following conditions: the axis of the cylinder is mapped to $\bar{\bar{K}}$; the top and bottom of the cylinder correspond to planar surfaces of which one contains the sub-segment $\bar{e}\bar{e}'$ of $\bar{e}p_3(0)$, the other contains the segment $\bar{\bar{e}}\bar{\bar{e}}'$ of $\bar{\bar{e}}p_3(0)$; Z shall be distinct from the segments $\bar{e}p_3(0)$ and $\bar{\bar{e}}p_3(0)$ except for $\bar{e}\bar{e}'$ and $\bar{\bar{e}}\bar{\bar{e}}'$. Furthermore Z shall be so close to $\bar{\bar{K}}$ that Z excludes the curve $\bar{K}_3''$. Hence, in particular, $\bar{e}'$ lies between $\bar{e}$ and $\bar{e}''$, $\bar{\bar{e}}'$ lies between $\bar{\bar{e}}$ and $\bar{\bar{e}}''$. Since $\bar{K}_3$ can be continuously deformed in $\bar{\bar{K}}$ within K, there exists a double-point free curve $\bar{K}_3'$ on Z with the following properties: $\bar{K}_3'$ connects $\bar{e}'$ with $\bar{\bar{e}}'$ and is distinct from $\bar{e}\bar{e}'$ and $\bar{\bar{e}}\bar{\bar{e}}'$ except for the endpoints; $\bar{K}_3'$ bounds together with $\bar{\bar{K}}$ and the segments $\bar{e}\bar{e}'$ and $\bar{\bar{e}}\bar{\bar{e}}'$ a singularity free disk E_1 lying in the interior of Z; $\bar{K}_3'$ can be deformed to $\bar{K}_3''$ within the exterior of Z in such a way that the endpoints move along $\bar{e}\bar{e}'$ respectively $\bar{\bar{e}}\bar{\bar{e}}'$ and such that no further intersection points with K occur under the deformation. Therefore in the interior of E, $\bar{K}_3'$ bounds together with $\bar{K}_3''$ and the segments $\bar{e}\bar{e}'$ and $\bar{\bar{e}}\bar{\bar{e}}'$ a disk (not necessarily singularity free) E_2 which is distinct from K except for the two boundary segments $\bar{e}\bar{e}'$ and $\bar{\bar{e}}\bar{\bar{e}}'$.

Let $\bar{\chi}$ and $\bar{\bar{\chi}}$ have the same meaning as in the previous paragraph. Let $\bar{K}_1$ be the image of the part of K defined by $\bar{\chi} \leq \chi \leq \bar{\bar{\chi}}$ under map (2). Since the part of K defined by $0 \leq \chi \leq \bar{\chi}$ as well as the part defined by $\bar{\bar{\chi}} \leq \chi \leq 1$ is mapped by (2) to the point $p_1(0)$, $\bar{K}_1$ is essentially identical to K_1. $\bar{K}_1$ (as well as K_1) be contracted in K to $p_1(0)$ fixing $p_1(\bar{\chi}) = p_1(\bar{\bar{\chi}}) = p_1(0)$. Let $\bar{K}_1''$ be a parallel curve to $\bar{K}_1$ defined
p. 670 in the following way: Let $\mu(\chi)$, $\bar{\chi} \leq \chi \leq \bar{\bar{\chi}}$ be a continuous non-negative function such that $\mu(\bar{\chi}) = \mu(\bar{\bar{\chi}}) = 0$; $\mu(\chi)$ shall be positive for the values of χ different from $\bar{\chi}$ and $\bar{\bar{\chi}}$; the maximum μ of $\mu(\chi)$ shall be smaller than the smallest of the distances $p_1(\chi)\bar{p}_3''(\chi)$ and small than the distance between $p_1(\chi)$ and each intersection point with K lying on a secant $p_1(\chi)p_3(\chi)$. Furthermore μ shall be so small that the set of points whose distance from K is at most equal to μ is still a one-to-one continuous image of a solid torus. Let $\bar{K}_1''$ be the set of points $p_1(\chi)''$ which lie beyond $p_1(\chi)$ on the segment $p_1(\chi)p_3(\chi)$ at at distance $\mu(\chi)$. Obviously $\bar{K}_1''$ bounds together with $\bar{K}_3''$ and the segments $p_1(0)\bar{e}''$ and $p_1(0)\bar{\bar{e}}''$ a disk E_3 which is swept out by the segments $p_1''(\chi)\bar{p}_3''(\chi)$ for $\bar{\chi} \leq \chi \leq \bar{\bar{\chi}}$. Firstly E_3 has the two segments $p_1(0)\bar{e}''$ and $p_1(0)\bar{\bar{e}}''$ belonging to the boundary in common with K and secondly, E_3 and K have r-intersection points in common which are intersection points of secants $p_1(\chi)p_3(\chi)$ with K. The boundary of E_3 has a double point in $p_1(0)$ by which the boundary is split into two parts $\bar{K}_1''$ and $p_1(0)\bar{e}'' + \bar{K}_3'' + \bar{\bar{e}}''p_1(0)$. One of these two parts, namely $\bar{K}_1''$, bounds a disk E_4 which has only the boundary point $p_1(0)$ in common with K. In order to show this, we interpret a line $\mathcal{G}$ with a solid cylinder $\mathcal{Z}$ (unbounded on both sides) as a universal covering space of the closed polygon K together with a torus neighborhood U of K. $\bar{K}_1''$ shall be completely contained in U. The curve $\bar{K}_1''$ corresponds to a system of mutually equivalent closed curves in $\mathcal{Z}$ since $\bar{K}_1$ can be contracted to the point $p_1(0)$ in K, hence also $\bar{K}_1''$ can be contracted to the point $p_1(0)$ in U. Let $\bar{\mathcal{R}}_1''$ be such a curve in $\mathcal{Z}$, $\mathfrak{p}_1(0)$ the point of $\bar{\mathcal{R}}_1''$ on $\mathcal{G}$ which corresponds to the point $p_1(0)$ of $\bar{K}_1''$. $\bar{\mathcal{R}}_1''$ has - except for $\mathfrak{p}_1(0)$ - no point in common with $\mathcal{G}$. The collection of all segments $\mathfrak{p}_1(0)\bar{\mathfrak{p}}_1''$ therefore spans (while $\bar{\mathfrak{p}}_1''$ traverses the curve $\bar{\mathcal{R}}_1''$) a disk $\mathcal{E}_4$ bounded by $\bar{\mathcal{R}}_1''$, which is distinct from $\mathcal{G}$ besides the boundary point $\mathfrak{p}_1(0)$. The disk $\mathcal{E}_4$ corresponds to a disk E_4 in U which is bounded by $\bar{K}_1''$ and which has only the boundary point $p_1(0)$ in common with K.

By combining the disks E_1, E_2, E_3, E_4 one obtains a disk E bounded by K which has exactly r boundary singularities. Indeed: E_1 is bounded by $\bar{e}\bar{e}' + \bar{\bar{K}} + \bar{\bar{e}}\bar{\bar{e}}' + \bar{K}_3'$ and possesses no intersection points with K. If one glues E_1 and E_2 along $\bar{K}_3'$, then one obtains a disk $E_1 + E_2$, which is bounded by $\bar{e}\bar{e}'' + \bar{\bar{K}} + \bar{\bar{e}}\bar{\bar{e}}'' + \bar{K}_3'$ and which also has no intersection points with K. If one glues the disk E_3 to E_1 and E_2 along $\bar{K}_3'$, then $E_1 + E_2 + E_3$ is bounded by $p_1(0)\bar{e} + \bar{\bar{K}} + \bar{\bar{e}}p_1(0) + \bar{K}_1''$, hence by $K + \bar{K}_1''$ and the number of intersection points with K is exactly equal to r. $E_1 + E_2 + E_3$ can be interpreted as a disk bounded by K with a hole bounded by $\bar{K}_1''$. The boundary p. 671
$\bar{K}_1''$ of the hole touches the boundary of K of the disk in the point $p_1(0)$. The hole is finally filled by pasting the disk E_4 (which has only the boundary point $p_1(0)$ in common with K) bounded by $\bar{K}_1''$ into $\bar{K}_1''$. New boundary singularities are not created.

Finally it remains to show that the disk E can be approximated by a triangulated disk in general position without increasing the number of boundary singularities: E_1 can be triangulated from the start; the approximation of E_2 by a disk which has also only the two boundary segments $\bar{e}'\bar{e}''$ and $\bar{\bar{e}}'\bar{\bar{e}}''$ in common with K, and the approximation of E_4 by a disk which intersects K only in the boundary point $p_1(0)$, do not created any difficulties. In order to approximate E_3 without increasing the number of intersection points with K, one has to know - as in the proof of the previous lemma - that the intersection points of K with E_3 are simple. For the proof of simplicity of the intersection points we repeat exactly the same arguments as in the proof as in Lemma 15. □

Lemma 17. *The set M_{13} contains at least k-double-point free arcs $C_1, C_2, \ldots, C_k$ satisfying the following conditions*

(1) C_ν $(\nu = 1, 2, \ldots, k)$ *contains no pair of points equivalent under a deck transformation except for the end points.*
(2) *The endpoints of each C_ν correspond to each other under the deck transformation $t_{\pm 1}$.*
(3) *If j-arcs (belonging to j different C_ν or to curves equivalent to one of the C_ν's) pass through a point x, y of M_{13}, then x, y is at least a j-tuple point of the set M_{13}.*

For the proof, one proceeds as in the proof of Lemma 10. There we only used

1. that the set $M(AB, A)$ has at least as many intersection points in common with each arc connection two points of the lines $y = 0$ and $y = 1$ as the number of curves C_ν that have to be constructed,
2. that from each complementary area of $M(AB, A)$ each point of the boundary can be reached,
3. and that in each component of this boundary each pair of points can be connected by an arc.

In contrast, the fact that $M(AB, A)$ is closed has not been used (which does not hold for M_{13}). The three previously mentioned properties (which are essential to the proof) hold for the set M_{13}, if one uses the Hausdorff notion of connectedness[23]
for the definition of a component. According to this, the conclusions of the proof p. 672
of Lemma 10 can also be applied here. Thus Lemma 16 plays the role of Lemma 9. The only modification is that the set G of points which can be connected with the

[23]Compare F. Hausdorff, Grundzüge der Mengenlehre (1914), p. 244.

line $y = 0$ with respect to M_{13} is now not necessarily a connected component. In this case for C_1^* we have to take the component which separates $y = 0$ from $y = 1$ of the set of points of M_{13} which can be reached from G.

Proof. of Theorem 2: The proof of the claim that the intersection M_{123} of M_{12} and M_{13} contains at least k^2 points - and therefore the proof of Theorem 2 - follows from Lemmas 17 and 15. Each of the k arcs in M_{13} chosen according to Lemma 17 intersects the set M_{12} in at least k points by Lemma 15, so that one in fact obtains k^2 intersection points of M_{12} and M_{13}. The distinctness of the intersection points in the described sense of multiplicity results from - as in the proof of Theorem 1 - the specific properties of the C_ν. □

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