Folds and cuts: mathematics and origami

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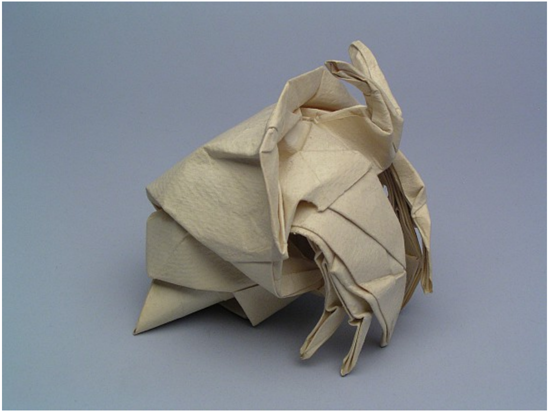
Origami

Origami objects can be 3D, but today's talk is about 2D origami.



Hermit Crab, opus 62 by Robert Lang (1986).

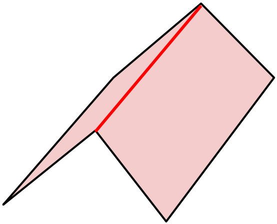
Folds

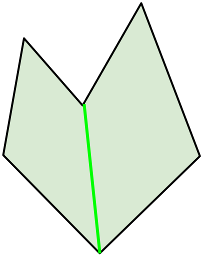
A *flat origami model* can be pressed onto the plane without introducing new creases.

Flat folds: flattened parallel layers of paper.

Folding paper creates one of two kinds of *creases*:

- Mountain (red)
- Valley (green)

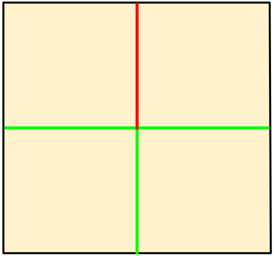


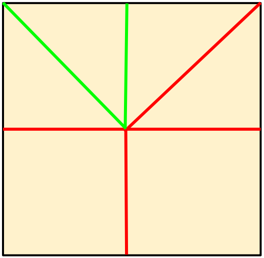


Single-Vertex Flat Origami Models

- A *vertex* is any point (not on the boundary) at which two or more creases meet.
- *Single-vertex flat folds* — the simplest origami construction.
- Let's play — do these *crease patterns* fold flat?

red = mountain fold, green = valley fold

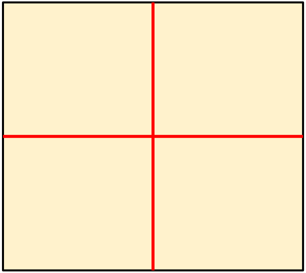


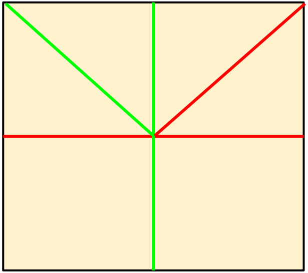


Can these crease patterns fold flat?

Any conjectures?

red = mountain fold, green = valley fold



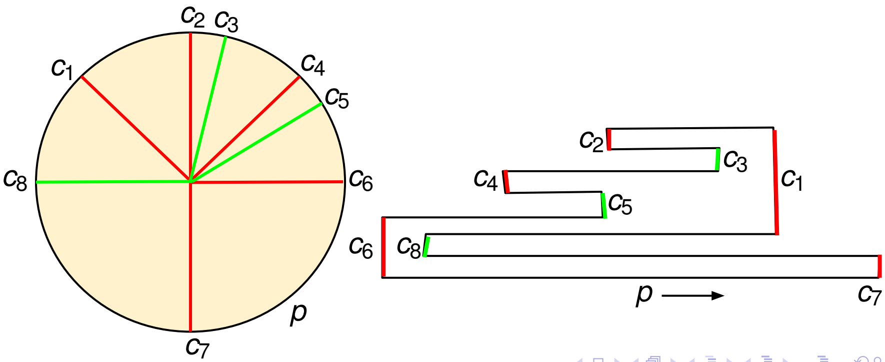


Maekawa-Justin theorem

Theorem (M-J)

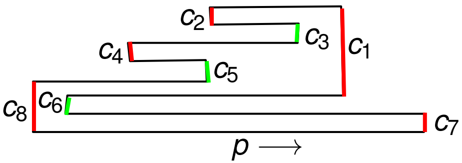
If M mountain and V valley creases meet at a flat vertex fold, then $M - V = \pm 2$.

Take flat origami model and snip near vertex.
Get a series of boundary arcs zig-zagging between creases.



Maekawa-Justin Proof 1:

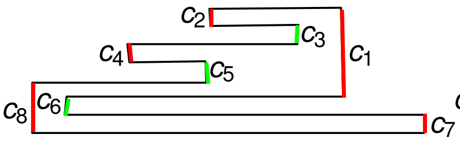
- Pick a non-crease point p on boundary, and walk to the right along boundary.
- Consider direction vector. Starts as 0.
- Mountain fold rotates direction vector $+\pi$.
- Valley fold rotates direction vector $-\pi$.
- Angle has changed by 2π when we return to p.
- $M \cdot (+\pi) + V \cdot (-\pi) = 2\pi$.
- Hence $M - V = 2 \quad$ (or $V - M = 2$). $\square$

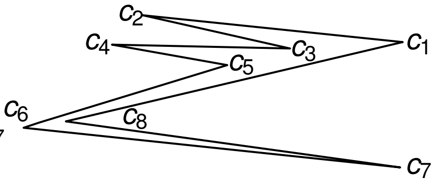


Maekawa-Justin Proof 2:

Recall: Total internal angle sum of an n-gon is $(n-2)\cdot\pi$.

- View boundary path as a 'squashed' polygon.
- Mountain vertices have internal angle of 0.
- Valley vertices have internal angle of 2π.
- Total internal angle sum: $M\cdot 0 + V\cdot(2\pi) = (n-2)\cdot\pi$.
- Creases correspond to vertices, so $n = M + V$.
- Altogether: $V\cdot(2\pi) = (M+V-2)\cdot\pi$.
- Rearranging: $2V = M + V - 2$, or $M - V = 2$. □





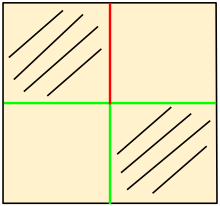
Theorem (Even degree)

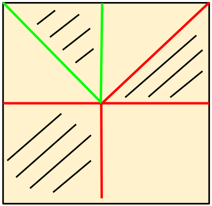
A vertex in a flat folding has even degree

Proof 1: If n is total # creases, M # mountain, V # valley; by Maekawa-Justin Theorem,
$n = M + V = 2V + M - V = 2V \pm 2 = 2(V \pm 1)$.

Feeling bored?

Proof 2: Prove that any flat origami model is 2 colorable. (Here the boundaries of the regions are the creases used in the final figure.) This immediately gives the result. □



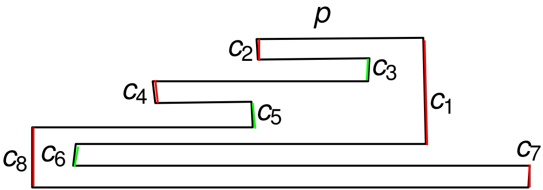


Theorem (Even # layers)

The number of layers of paper near a flat vertex fold at any point p (that does not intersect an edge) is even.

Proof:

- The point p is crossed once for every layer of paper.
- Every time it is crossed to the left, it must also be crossed to the right (to get back to the start).
- Have left-right pairs, hence an even number of layers. □

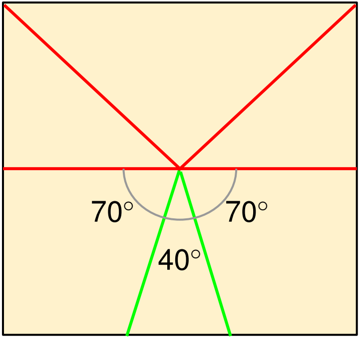


Note: Take L-R sequence. Its length is even (why?), & is degree of vertex! Gives third proof of even degree theorem

What about angles?

M-J Theorem holds for this crease pattern: $M - V = 4 - 2 = 2$.

Can this crease pattern fold flat?

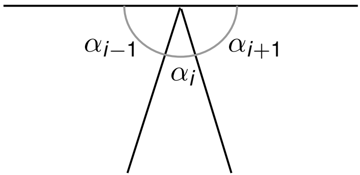


Big-Little-Big or Local Min Theorem

Theorem (Big-little-big)

Suppose that in a flat vertex fold we have a sequence of consecutive angles α_{i-1}, α_i and α_{i+1}, with $\alpha_i < \alpha_{i-1}$ and $\alpha_i < \alpha_{i+1}$. Then the two crease lines inbetween these angles cannot have the same mountain-valley parity.

Proof: Same parity means the larger angles cover up α_i on the same side of the paper.
Impossible (without the paper intersecting itself). □

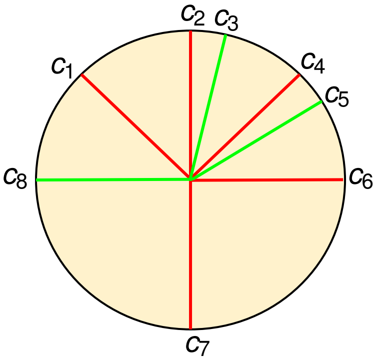


Kawasaki-Justin Theorem

Theorem (K-J)

A set of even creases meeting at a vertex v folds flat, if and only if, the alternating sum of the consecutive angles between the creases at v is zero:

$$\alpha_1 - \alpha_2 + \alpha_3 - \cdots - \alpha_{2n} = 0.$$



$\Longrightarrow$ (sketch)
Start at leftmost edge, track angle traveled w.r.t. vertex. Travel left, then right, repeat. Total angle traveled is 0°.

$\Longleftarrow$ construction with tricky cases. □

Corollary to Kawasaki-Justin

Corollary

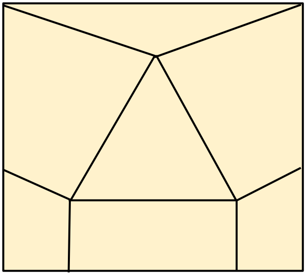
The angle sum between two consecutive creases of a flat vertex fold is always $\leq \pi$.

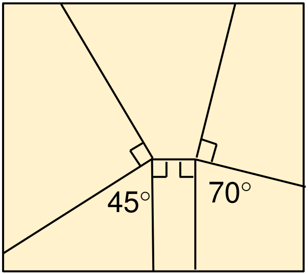
Proof:

- We know $\alpha_1 + \alpha_2 + \alpha_3 + \cdots + \alpha_{2n} = 2\pi$.
- K-J means $\alpha_1 - \alpha_2 + \alpha_3 - \cdots - \alpha_{2n} = 0$.
- Combining yields $\alpha_1 + \alpha_3 + \cdots + \alpha_{2n-1} = \pi$, and $\alpha_2 + \alpha_4 + \cdots + \alpha_{2n} = \pi$.
- Equality when vertex has degree 2: $\alpha_1 = \alpha_2 = \pi$. □

Flat folds with more than one vertex

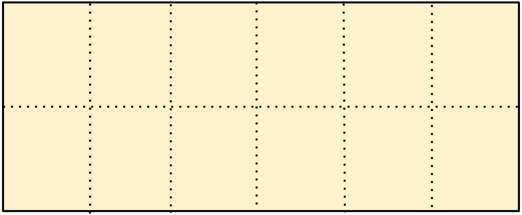
- What can go wrong?
- Mountain-Valley fold contradictions.
- Self-intersection of paper.





Flat Foldability is hard

- *Computational origami* studies these questions.
- Marshall Bern & Barry Hays (1990s): deciding whether a crease pattern is flat foldable is NP-hard (even with M & V specified).
- Bern-Hays proof fails for special case of folding *maps*.
- *Open problem:* Is there an efficient algorithm for deciding whether or not a given rectangular map can be folded flat? (Here M & V specified.) Even $2 \times n$ grids are unsolved!

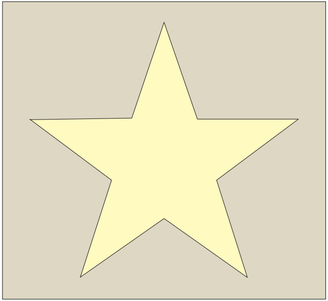


5-pointed star

Question

How many cuts are needed to create a regular 5-pointed star?

- Answer known to Houdini in his 1922 book *Paper Magic*.
- Answer known to Betsy Ross – she convinced George Washington to use this star on the American flag as it is easy to make.

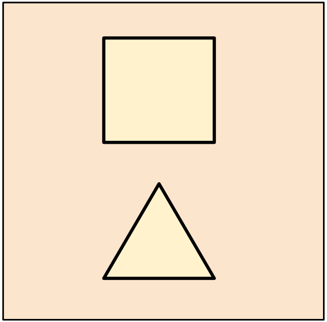


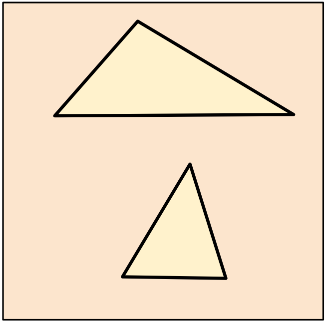
Fold and one-cut triangles and squares

Your turn: fold these shapes and then make one cut.

What do you notice?

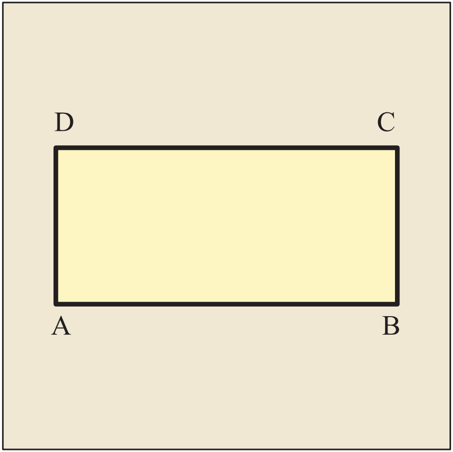
What fact about triangles do you notice?





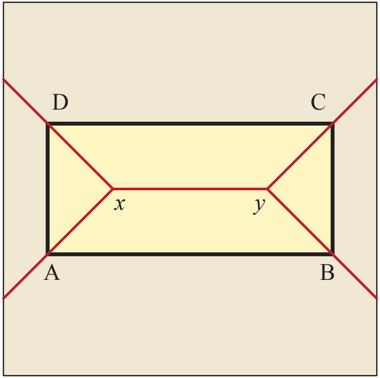
Fold and one-cut rectangles

There are (at least) two methods.



Fold and one-cut rectangles

- Need angle bisectors, plus central fold between x and y.
- Vertices do not satisfy Maekawa-Justin Theorem!
- Add valley folds at x and y *perpendicular* to *ABCD*'s edges.



Fold and One-Cut Theorem

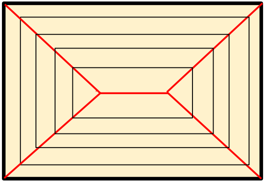
Theorem

Any straight-line drawing (one composed of straight segments) on a sheet of paper may be folded flat so that one straight scissors cut completely though the folding, cuts all the segments of the drawing and nothing else.

- Proof very tricky.
- First discovered by Erik Demaine, Marty Demaine, and Anna Lubiw (1999).
- Many cases not covered — proof repaired twice independently!
- Complete proof in Ch 17 *Geometric Folding Algorithms* by Erik Demaine and Joe O'Rourke.

Sketch of proof: fold and one-cut

- Construct *straight skeleton* of the shape.
  - Shrink shape down in size.
  - Skeleton traced out by path of vertices.
- This corresponds to creases needed.
- Add perpendiculars to every skeleton vertex.
- Show construction ‘works’!
- Why so many cases?
  - certain vertices may not need perpendiculars,
  - perpendiculars may ‘wander’ infinitely.



Crease pattern for a turtle

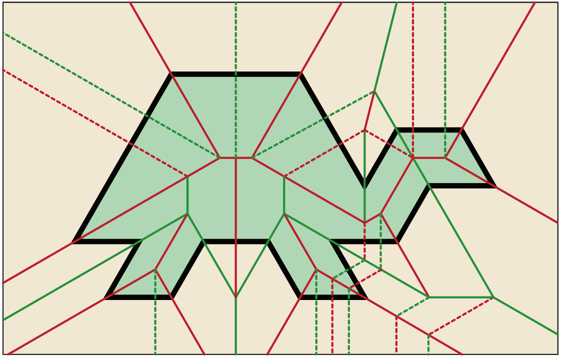


Figure by Joe O'Rourke

Crease pattern for the letter A

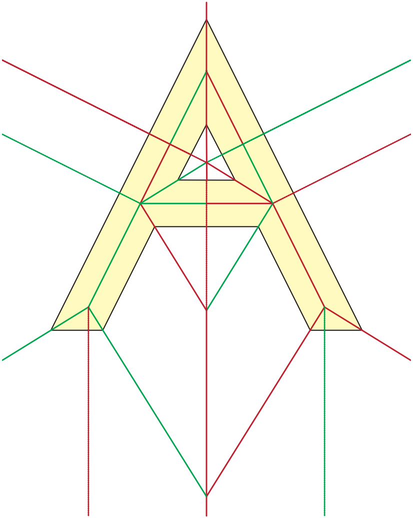
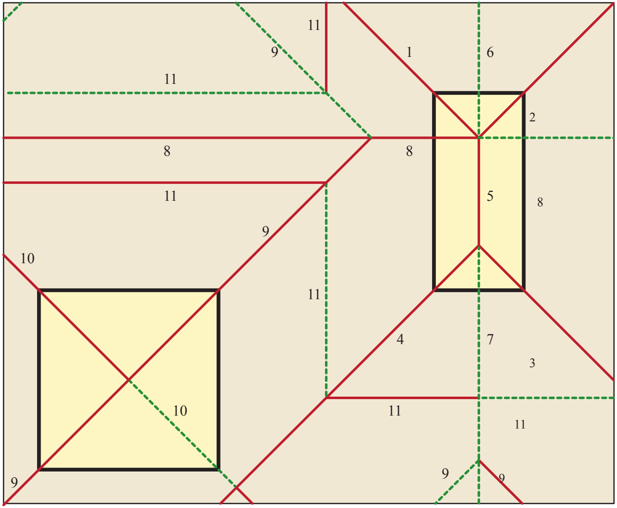


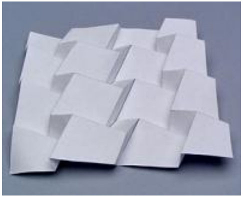
Figure by Joe O’Rourke

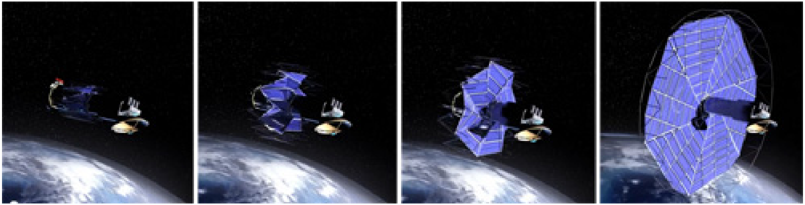
Crease pattern for a square and a rectangle



Origami is beautiful and useful.

Example: the Muira fold is rigid & moves in a prescribed way.
Used to store and unfold solar arrays in satellites (BYU)





Thank you!

Further Reading:

- *How to fold it: the mathematics of linkages, origami and polyhedra* by Joesph O'Rourke, 2011.
- *Project Origami: activities for exploring mathematics* by Thomas Hull, 2006.
- *Art & Sculpture: mathematical inquiry in the liberal arts* by Julian F. Fleron, Volker Ecke, Christine von Renesse, Philip K. Hotchkiss, 2013.
- *Origami Design Secrets* by Robert Lang, 2011.