Knots and Minimum Distance Energy

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Professor Elizabeth Denne and I continue work I started in a research program (summer 2007). We aim to find which polygonal knots have least Minimum Distance Energy. I previously showed that the energy is minimized for convex polygons. We hope relating the energy to chords of polygons will be a helpful step towards showing that regular n-gons have the least minimum distance energy for all polygonal knots.

1. INTRODUCTION

Minimum distance energy is an energy on polygonal knots which has recently been defined. Jonathan Simon [14] showed that the minimum distance energy of polygonal knots exists and investigated some of its properties in 1997. He found that in experiments seeking to find the position for least minimum distance energy of various polygons, edge lengths equalize and angles open to become closer to those of regular polygons. Experiments I did in my previous research [15] also suggested that as minimum distance energy is reduced knots open and equalize. Several works explore the relationship between the Möbius energy (also known as O'Hara energy) of smooth knots and the minimum distance energy of polygonal knots [11, 12, 16]. Möbius energy is minimized for a perfect circle [11, 12], and a regular n-gon is a polygonal approximation of a circle.

Based on this evidence, we conjectured that regular n-gons have least minimum distance energy. In my previous research I noted that a standard planar circle also encloses the most area for a fixed perimeter [15] and similarly, that the regular n-gon maximizes area for all n-gons [3]. Thus, I explored techniques used to solve the isoperimetric problem in order to build support for the conjecture that the regular n-gon minimizes minimum distance energy. That work allowed me to show that convex polygonal knots minimize this energy.

Elizabeth Denne and I have been investigating results by Lükő Gábor related to the average chord length of curves. Jason Cantarella [1] worked with others to use Lükő's results to prove that circles minimize Möbius Energy. We now investigate to see if similar methods can be used to show that the discrete version, minimum distance energy, will be minimized for regular n-gons, that is for polygonal approximations of the standard planar circle. We give advice on how Lükő's results could be manipulated to help prove the following conjecture.

Conjecture. Regular n-gons have least minimum distance energy.

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1.1. Definitions and theorems

Definition 1. We denote the length of a line segment, X, as $\ell(X)$. A line segment between the points p and q is denoted $\overline{pq}$ and has length $|p-q|$. The minimum distance between the line segments X and Y occurs between some point p on X and some point q on Y, and the distance $|p-q|$ is denoted $md(X,Y)$.

Definition 2 ([11])**.** The Minimum Distance Energy for a pair of nonconsecutive edges, X and Y, of an n-gon, is $U_{md}(X,Y) = \frac{\ell(X)\ell(Y)}{md(X,Y)^2}$. For a polygon P,

$$U_{md}(P) = \sum_{\text{all edges } X} \; \sum_{Y \neq X, \text{ nor adjacent}} U_{md}(X,Y).$$

Remark 1. Note that U_{md} is defined only for n-gons with $n \geq 4$ or else the n-gon would have no pairs of non-adacent edges.

Remark 2. Simon's original formula [14] for the Minimum Distance Energy of a polygon P, is

$$U'_{md}(P) = \sum_{X,\, Y, \text{ non-consecutive edges of } P} U_{md}(X,Y).$$

We will refer to U_{md}, as minimum distance energy although it is slightly different than this original version. The definitions differ as one defined for unordered the other for ordered pairs and therefore $U_{md}(P) = 2 \cdot U'_{md}(P)$, so any differences between the definitions are otherwise trivial. Following the example of [11], we use U_{md} since it can be thought of as the discrete version of Möbius Energy, defined below.

Definition 3 ([1, 11])**.** The *Möbius Energy* (or O'Hara Energy) of a smooth knot, K is

$$E_0(K) = \iint_{C\times C} \frac{1}{|x(t)-x(s)|^2} - \frac{1}{|s-t|^2}\, ds\, dt$$

where $t \to x(t)$ gives a unit-speed parameterization of K on a standard planar circle C. The notation, E_0, reminds us that we are using a definition of Möbius Energy such that $E_0(C) = 0$, for the standard planar circle, C.

Minimum distance energy is a function, which always has a real output since it is just the addition, division and multiplication of distances which are given by real values. Therefore it is natural to aks whether or not U_{md} can be minimized for polygons in a given knot class.

Many polygonal knots have crossings, that is, no matter how the edge lengths and angles are changed they cannot be made into a regular n-gon. We argue that regular n-gons, polygonal unknots, minimize U_{md}. The following theorems, given by Simon [14], suggest that unknots minimize U_{md}. Most importantly, Theorem 2 proves that there exists an n-gon which minimizes U_{md}.

Theorem 1 ([14])**.** *If $[K]$ is a knot type with q crossings and K is a polygonal representation of this knot type, then $U_{md}(K) \geq 4\pi q$.* □

Theorem 2 ([14])**.** *For each knot type $[K]$ represented by an n-segment polygon there exists a knot K_0 with n edges such that $U_{md}(K_0)$ is minimum amoung all n-segment polygons representing the knot type $[K]$.* □

Remark 3. Note that this theorem implies that we can find a knot which minimizes U_{md}. So, if regular n-gons minimize this energy, then they are examples of the minimizing knots, K_0, where $[K]$ represents all unknots. Additionally, Theorem 2 states that *there exists* at least one K_0 which minimizes the energy, but this does not suggest that there is only one such K_0.

Theorem 3 ([14])**.** *For segments of an n-gon denoted by $A_1,\ A_2,\ldots,A_n,\ md(A_i,A_{i+2}) \geq \ell(A_{i+1})$.* □

Theorem 4 ([16])**.** *The regular 4-gon minimizes U_{md} for all knots with 4 edges.* □

Lemma 5 ([14])**.** *If K is a polygon of n edges $(n \geq 5)$ then $U_{md}(K) \geq 2n$. Note the inequality is strict when $n \geq 6$.* □

Theorem 6 ([14])**.** *For each real number, r, the set of all polygons K having $U_{md}(K) \leq r$ represents only finitely many knot types.* □

Example 1. The regular 5-gon, R_5, has the nice property that the minimum distance between each pair of edges is given by the edge between them. Since U_{md} is scale invariant, it is sufficient to investigate the regular 5-gon $ABCDE$, with unit edge lengths, as found in Figure 1. Take the edge A. It has two non-adjacent edges C and D. The shortest distance between A and C is the same as the length of edge B. Similarly, $md(A,E) = \ell(E)$. Regular 5-gons have 5-fold symmetry, so,

$$U_{md}(R_5) = 5\left(\frac{\ell(A)\ell(C)}{md(A,C)} + \frac{\ell(A)\ell(D)}{md(A,D)}\right) = 5\left(\frac{1}{1}+\frac{1}{1}\right) = 5\cdot 2 = 10.$$

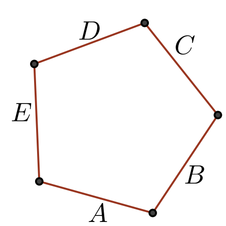

FIG. 1: For a regular pentagon the minimum distance between non-adjacent edges is given by the pentagon's edge lengths.

Example 2. There exists a non-regular 5-gon, call it W_5, which also has $U_{md}(W_5) = 10$ which can be found using the program Ming [18]. This does not disprove our Conjecture 1, however, because we want to show that regular n-gons have least minimum distance energy, and $U_{md}(W_5) = U_{md}(R_5)$ (we just calculated it), but it is not the case that $U_{md}(W_5) < U_{md}(R_5)$.

Example 3. It is important to notice that the minimum distance between two edges is not always the same as the length of another edge. For example, let us investigate the 4-gon, W_4, found in Figure 2. Let W_4 have vertices a, b, c and d and edges $\overline{ab} = A$, $\overline{bc} = B$, $\overline{cd} = C$ and $\overline{da} = D$. This 4-gon was constructed in $\mathbb{R}^2$, by defining $a = (0, 0)$, $b = (3, 0)$, $c = (4, 1)$, and $d = (1, 3)$. By the Pythagorean Theorem, $\ell(A) = 3$, $\ell(B) = \sqrt{2}\,\ell(C) = \sqrt{13}$, and $\ell(D) = \sqrt{10}$. By the Law of Sines, the $md(A, C) = md(C, A) \approx 1.39$ and $md(B, D) = md(B, D) \approx 2.85$ (each distance shown with green lines in Figure 2). Therefore, $U_{md}(W_4) \approx 2\left(\frac{3\sqrt{13}}{1.39}\right) + 2\left(\frac{\sqrt{2}\sqrt{10}}{2.85}\right) \approx 18.70$.

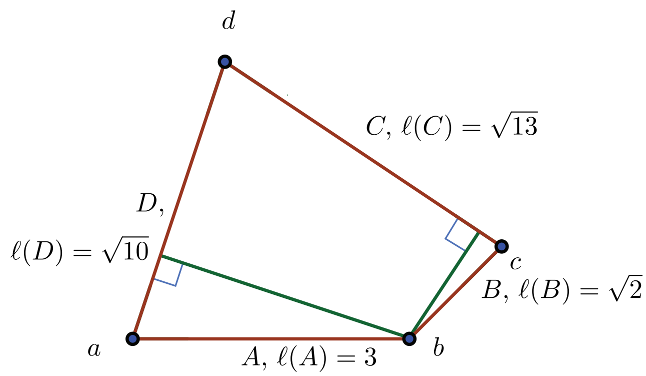

FIG. 2: Example polygon, W_4, with shortest distance between edges given in green.

1.2. General formulas for U_{md} of regular n-gons

Lemma 7. *For all non-adjacent edges X and Y in a regular n-gon, $md(X, Y)$ is the distance between a vertex of X and a vertex of Y.*

Proof. Observe that a regular n-gon has n-fold symmetry. Therefore, the minimum distances between vertices must also have symmetry. This could not be true if there were minimum distances which did not occur between vertices as in Figure 2. Note that if edges are parallel, there are other line segments that have the same minimum distance as those between vertices (this occurs in Figure 3, the the purple line is not a distance between vertices of the n-gon, however there exists a segment between vertices, shown in green, which has this same length). In such cases, since the minimum distance also occurs between vertices, the existence of parallel minimum distance paths (not necessarily between vertices) is trivial. □

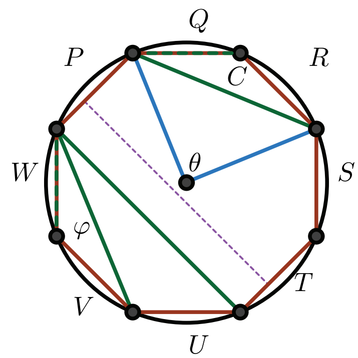

FIG. 3: Inscribing an octagon in a circle allows us to find distances between vertices. Minimum distances from P are shown by the green edges.

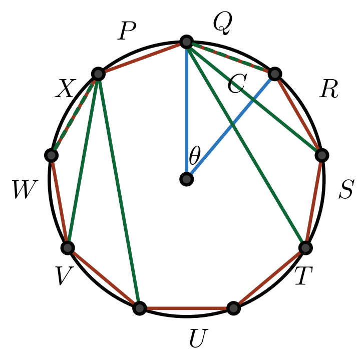

FIG. 4: Inscribing a 9-gon in a circle also allows us to find minimum distances. Minimum distances from P are shown by the green edges.

One can inscribe a regular n-gon on a circle as in Figures 3 and 4. By Lemma 7 the minimum distances between edges are the lengths of chords of a circle. The length of a chord, C, of a circle with radius r is $\ell(C) = 2r\sin(\frac{\theta}{2})$, where θ subtends C. In our case $r = 1$ and side lengths of the n-gon are given by $\theta = \frac{2\pi}{n}$. That is, sides have length $2\sin(\frac{\pi}{n})$.

In the regular 8-gon shown in Figure 3, the green paths give the minimum distances between the edge P and all edges ($R,\ S,\ T,\ U$, and V) to which R is not adjacent. Note that these distances are all chords of a circle. For the regular octagon shown, $md(P, U)$ is calculated by $\theta = 2\frac{2\pi}{8}$, $md(P, T)$ has $\theta = 3\frac{2\pi}{8}$, and so on.

Hence, for n-gons with n even, if we fix an edge X and we find:

$$\sum_{\text{all edges} X \neq Y, \text{ nor adjacent}} U_{md}(X, Y) = \frac{\sin^2(\frac{\pi}{n})}{\sin^2\left(\frac{\pi(n-2)}{2n}\right)} + 2 \cdot \sum_{j=1}^{\frac{n}{2}-2} \frac{4\sin^2(\frac{\pi}{n})}{4\sin^2(\frac{\pi j}{n})}.$$

The minimum distance energy of the n-gon will be n times this sum, by the symmetry of regular n-gons. We can then simplify to obtain the formula given below.

$$U_{md}(R_n) = n \cdot \sin^2\left(\frac{\pi}{n}\right)\left(\frac{1}{\sin^2\left(\frac{\pi(n-2)}{2n}\right)} + 2 \cdot \sum_{j=1}^{\frac{n}{2}-2} \frac{1}{\sin^2\left(\frac{j\pi}{n}\right)}\right).$$

For n-gons with n odd, note that there are no parallel sides and there is an even number of minimum distance paths from each edge.

Thus, the minimum distance energy of R_n is:

$$U_{md}(R_n) = 2n \cdot \sin^2\left(\frac{\pi}{n}\right) \sum_{j=1}^{\lfloor\frac{n}{2}\rfloor-1} \frac{1}{\sin^2\left(\frac{j\pi}{n}\right)}.$$

Example 4. By formula for even n,

$$\begin{aligned} U_{md}(R_4) &= 4 \cdot \sin^2\left(\frac{\pi}{4}\right)\left(\frac{1}{\sin^2\left(\frac{\pi(4-2)}{2\cdot 4}\right)} + 2 \cdot \sum_{j=1}^{\frac{4}{2}-2} \frac{1}{\sin^2\left(\frac{j\pi}{4}\right)}\right) \\ &= 4 \cdot \sin^2\left(\frac{\pi}{4}\right)\left(\frac{1}{\sin^2\left(\frac{2\pi}{8}\right)} + 0\right) \\ &= 4 \end{aligned} \tag{1}$$

You may notice that this is much smaller than the value of $U_{md}(W_4)$, calculated above. This also supports our Conjecture.

2. PREVIOUS RESULTS

Relating U_{md} to the area inside a polygonal unknot, I used tools developed to give various solutions to the isoperimetric problem [2, 3, 5, 7, 8, 10, 19] to obtain the result that U_{md} is minimized for convex n-gons. First I proved for all planar polygons, convex polygons with the same edge lengths minimize U_{md} (Theorem 10). Then I proved a more general theorem which expands this result to all polygonal chains in $\mathbb{E}^3$ (Theorem 12).

2.1. The planar result

Definition 4. The *convex hull* of an n-gon, P, is the smallest convex set containing all vertices of P and is denoted $H(P)$. The boundary is denoted $\partial H(P)$.

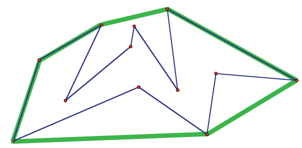

FIG. 5: The convex hull (green) of a non-convex polygon (blue).

We proved that the reflection of edges inside the convex hull of a polygon over the boundary of the convex hull minimizes U_{md} (Lemma 9). However some polygons, such as the one shown in Figure 7, require many reflections before they become convex. So, it was also necessary to show that all polygons become convex after a finite number of reflections. Fortunately, we found that this problem was first proposed by the famous mathematician Paul Erdös in 1934, and then solved by Bèla Nagy in 1939 [4, 17]. Toussaint gives a history and summary of related contemporary problems in [17]. We borrow his terms "flip" "pocket," and "pocket lid." These terms are defined below and illustrated in Figure 6.

Definition 5 ([17])**.** A *pocket* is a set of edges of a polygon not in $\partial H(P)$ between the vertices i and j on $\partial H(P)$. Its *pocket lid* is the line segment $\overline{ij}$.

Definition 6 ([17])**.** A *flip* is the reflection of a pocket across a pocket lid.

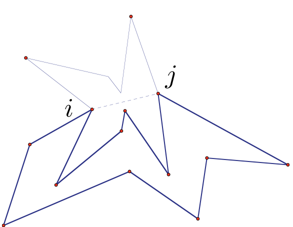

FIG. 6: Flipping a pocket over its pocket lid, $\overline{ij}$.

Theorem 8 (Erdös-Nagy [4, 17])**.** *Every simple planar polygon can be made convex with a finite number of flips.* □

Lemma 9 ([15])**.** *Let P be a non-convex simple planar polygon and P' the result of a flip on P, then $U_{md}(P) \geq U_{md}(P')$.*

Proof. Take a pocket, p of P. We shall refer to $P - p$ as p'. Perform a flip on p. We denote the reflected collection of edges as r, and the new polygon created as P'.

Edge length is not changed by reflections, so $\ell(X) = \ell(X')$, where X' is a corresponding edge in r. Therefore $\ell(X)\ell(\Delta) = \ell(X')\ell(\Delta)$, where Δ is a non-adjacent edge in p'. Thus, we need only show that $md(X', \Delta) \geq md(X, \Delta)$ in order to show $\frac{\ell(X')\ell(\Delta)}{md(X',\Delta)^2} \leq \frac{\ell(X)\ell(\Delta)}{md(X,\Delta)^2}$.

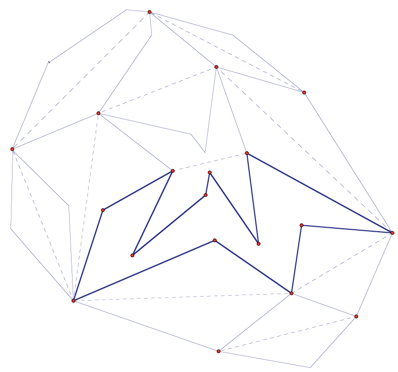

FIG. 7: A non-convex polygon with pockets flipped across pocket lids, creating another non-convex polygon. Flips are continued until a convex polygon is made.

Two nonadjacent edges in P' will either be on opposite sides of $\overline{ij}$ or on the same side.

Case 1 For the pairs (X, Y) on the same side of $\overline{ij}$, the edges X and Y are reflected together so $\frac{\ell(X')\ell(Y')}{md(X',Y')^2} = \frac{\ell(X)\ell(Y)}{md(X,Y)^2}$. The position of edges in p' is not affected by the flip, so distances between these edges will also remain the same after the flip. Thus, the U_{md} of pairs of nonadjacent edges on the same side of $\overline{ij}$ is unchanged.

Case 2 Now lets us investigate the distance between nonadjacent edges which are on opposite sides of $\overline{ij}$ after the flip. Let us say that the minimum distance, $md(X, \Delta)$, occurs between some point δ in Δ and some point $\alpha \in X$. Similarly, $md(X', \Delta)$ occurs between some δ' in Δ and some $\beta \in X'$. If we divide the plane along the pocket lid, $\overline{ij}$, we see that P is all on oneside of $\overline{ij}$, by definition of convex hull. All of r, however, is on the opposite side of $\overline{ij}$ (although i and j are on the line). Thus, for all δ and β, $|\delta - \beta| \geq |\delta - \alpha|$ which implies $\frac{\ell(X')\ell(\Delta)}{md(X',\Delta)^2} \leq \frac{\ell(X)\ell(\Delta)}{md(X,\Delta)^2}$.

Hence, $U_{md}(P') \leq U_{md}(P)$. □

Theorem 10 ([15]). *If P is a planar n-gon with minimized U_{md}, then P is convex.*

Proof. By Lemma 9, this flipping process minimizes U_{md}. By applying Theorem 8, we know that a finite number of flips creates a convex polygon. Therefore, the convex polygon minimizes U_{md} for planar polygons. □

2.2. Polygonal curves in $\mathbb{E}^3$

G. T. Sallee [13] "stretches" polygonal curves in $\mathbb{E}^n$, a process which can increase the distance between points on different edges by changing angles, not edge lengths. He finds that any polygonal curve in $\mathbb{E}^n$ can be made planar and convex with a finite number of stretches [13]. We concern ourselves, here, with only 3 dimensions. Supported by his work, we prove that the planar convex n-gon minimizes U_{md} for higher dimensions (Theorem 12).

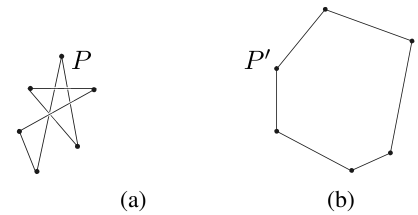

FIG. 8: (b) A convex polygon made by stretching the polygon in (a).

Definition 7 ([13])**.** A *stretch* is made by a change in angles. For P and P', polygons with corresponding lengths, P' is a *stretched* version of P, if for all $x,\ y\ \in P$ and corresponding $x',\ y'\ \in P'$, $|x-y| \leq |x'-y'|$. □

Lemma 11 ([13])**.** *If P is a non-convex polygon in $\mathbb{E}^n$, there exists a stretched polygon, P' which is planar and convex, such that for all points $x,\ y \in P$, with x and y not on the same edge of P, and corresponding $x',\ y' \in P'$, then $|x-y| < |x'-y'|$.* □

Theorem 12 ([15])**.** *If P is a polygon in $\mathbb{E}^3$ there exists a convex planar polygon, P', created by stretching such that $U_{md}(P) \geq U_{md}(P')$.*

Proof. Let P be any polygon in $\mathbb{E}^3$ and P' the stretched convex planar polygon guaranteed by Lemma 11. Let X be an edge in P and X' be the corresponding edge in P'. Stretching does not change edge lengths. Therefore, again, we need only examine the minimum distances between two edges in each polygon. By Lemma 11, we know that for all points $x \in X$ and $y \in Y$, and corresponding x' and $y' \in P'$, $|x-y| < |x'-y'|$. Thus, $md(X,Y) \leq md(X',Y')$, and $\frac{\ell(X')\ell(Y')}{md(X',Y')^2} \leq \frac{\ell(X)\ell(Y)}{md(X,Y)^2}$. □

3. NEW INVESTIGATIONS

We now attempt to build evidence supporting the idea that regular n-gons minimize U_{md} with new methods. Lükő's theorems [6] give both discrete and continuous results relating to average chord length of curves and polygonal curves. We attempt to manipulate his results to prove our conjecture. Perhaps most exciting for our research is Lükő's Theorem II. When $g(t) = t$, this theorem implies the average squared distance between the vertices of an n-gon is maximized by the regular n-gon. It is promising that Cantarella [1] and others tweak Lükő's results to show that Möbius Energy is minimized on perfect circles, (the continuous version of our conjecture about regular n-gons). We were able to apply Lükő's Theorem II, to show that 5-gons have least U_{md} for all equilateral n-gons. In general, it seems that his theorem could be applied in order to show that regular n-gons minimize U_{md} for all equilateral n-gons, as will be explained. However, we could not directly apply Lükő's result to make any stronger conclusions.

Definition 8. Let the vertices of a n-gon be labeled $1, 2, \ldots, n$ and let $r_{i,l}$ denote the distance between vertices i and $i + l$. Let a be a constant greater than or equal to the length of every edge (denoted $r_{i,1}$) of the n-gon.

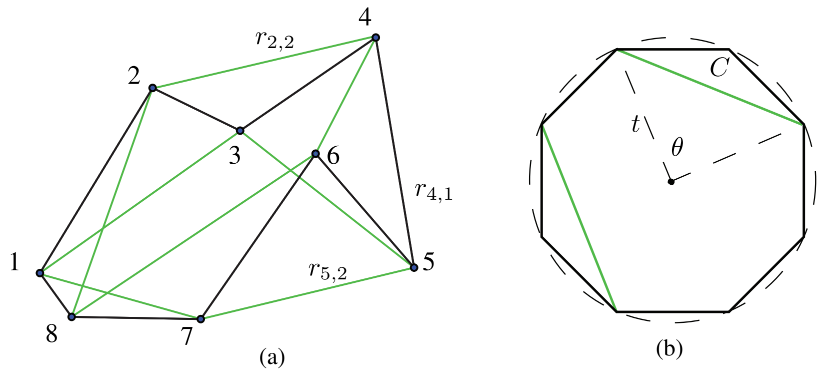

FIG. 9: (a) Polygon with edges ($r_{i,1}$) in black, each $r_{i,2}$ is given in green. (b) Regular octagon inscribed in circle of radius t; distances between vertices are the length of a chord $\ell(C) = 2t\sin(\frac{\theta}{2})$.

Definition 9. Let P and Q be points on a closed curve C, $r_{P,Q}$ the distance between them, s_P and s_Q, arc-length parameters and let L be the length of C.

Definition 10. An *affine image is the result of a transformation made of any composition of rotations, translations, dilations or shears.*

Theorem 13 ([6] Theorem I)**.** *If $g(t)$ is an increasing, concave function, the integral*

$$\frac{1}{L^2}\int_C\int_C g(r_{P,Q})ds_P ds_Q$$

attains its maximum only at circles. □

Theorem 14 ([6] Theorem II)**.** *Let $g(t)$ be an increasing, concave function, then,*

$$\frac{1}{n}\sum_{i=1}^{n} g(r_{i,l}^2) \leq g\left(a^2\frac{\sin^2(\frac{l\pi}{n})}{\sin^2(\frac{\pi}{n})}\right)$$

for all $n \geq 4$, with equality if and only if the n-gon is regular. □

Theorem 15 ([6] Theorem III)**.** *For any n-gon*

$$\sum_{i=1}^{n} r_{i,l}^2 \leq \left(\frac{\sin^2(\frac{l\pi}{n})}{\sin^2(\frac{\pi}{n})}\right)\sum_{i=1}^{n} r_{i,1}^2$$

where equality holds if and only if the n-gon is the affine image of a regular n-gon. □

When $g(t) = t$, Theorem 14 implies the average squared distance between the vertices of an n-gon is maximized by the regular n-gon. While the minimum distance between to edges in an n-gon may not necessarily occur between vertices, it is less than or equal to the distance between vertices. If we over estimate the minimum distance for non-regular n-gons we will find a U_{md} which is less than or equal to U_{md} for the actual n-gon. If the U_{md} of a regular n-gon with the same number of sides is less than or equal to the under approximation of U_{md} for a non-regular n-gon it implies regular n-gons minimize U_{md}.

Applying Theorem 14 for $g(t) = t$ for equilateral n-gons we obtain:

$$\frac{1}{n}\sum_{i=1}^{n} r_{i,l}^2 \leq \frac{\sin^2(\frac{l\pi}{n})}{\sin^2(\frac{\pi}{n})} \tag{2}$$

The minimum distances between edges X and Y a regular n-gon is:

$$md(X,Y) = \left(2\sin\left(\frac{\pi}{n}\right)\right)\left(\sin\left(\frac{l\pi}{n}\right)\right).$$

Therefore:

$$\begin{aligned} U_{md}(X,Y) &= \frac{(2\sin(\frac{\pi}{n})\sin(\frac{\pi}{n}))^2}{(2\sin(\frac{\pi}{n})(\sin(\frac{l\pi}{n}))^2} \\ &= \frac{(\sin(\frac{\pi}{n}))^2}{(\sin(\frac{l\pi}{n}))^2}. \end{aligned} \tag{3}$$

Inequality (2) implies:

$$\frac{1}{\frac{1}{n}\sum_{i=1}^{n} r_{i,l}^2} \geq \frac{1}{\frac{\sin^2(\frac{l\pi}{n})}{\sin^2(\frac{\pi}{n})}} \tag{4}$$

which can be simplified to:

$$\frac{n}{\sum_{i=1}^{n} r_{i,l}^2} \geq \frac{(\sin(\frac{\pi}{n}))^2}{(\sin(\frac{l\pi}{n}))^2}.$$

Multiplying both sides by n implies:

$$\frac{n^2}{\sum_{i=1}^{n} r_{i,l}^2} \geq n\frac{(\sin(\frac{\pi}{n}))^2}{(\sin(\frac{l\pi}{n}))^2}. \tag{5}$$

However, we seek to show that:

$$\sum_{i=1}^{n} \frac{1}{r_{i,l}^2} \geq n \frac{(\sin(\frac{\pi}{n}))^2}{(\sin(\frac{l\pi}{n}))^2}$$

in order to prove that regular n-gons minimize U_{md} for all equilateral n-gons. However, you will note that the left hand here does not necessarily equal the left hand in (5). If it can be shown that

$$\sum_{i=1}^{n} \frac{1}{r_{i,l}^2} \geq \frac{n^2}{\sum_{i=1}^{n} r_{i,l}^2}$$

then

$$\sum_{i=1}^{n} \frac{1}{r_{i,l}^2} \geq n \frac{(\sin(\frac{\pi}{n}))^2}{(\sin(\frac{l\pi}{n}))^2}.$$

However, we were not able to find computations that show the previous two inequalities are true.

Theorem 16. *For all equilateral 5-gons,* E_5*,*$U_{md}(R_5) \leq U_{md}(E_5)$*.*

Proof. Without loss of generality, let the length of each side be 1. Label the sides of E_5 : A, B, C, D, and E. Then

$$U_{md}(E_5) = 2\left(\frac{1}{md(A,C)^2} + \frac{1}{md(A,D)^2} + \frac{1}{md(B,D)^2} + \frac{1}{md(B,E)^2} + \frac{1}{md(C,E)^2}\right).$$

Since, by Theorem 3 the least minimum distance between two edges X and Y is $md(X,Y) = 1$, so $U_{md}(X,Y) = \frac{1}{md(X,Y)^2}$ is maximized when $md(X,Y)^2 = 1^2 = 1$.

Therefore, $U_{md}(E_5) \geq 2(\frac{1}{1} + \frac{1}{1} + \frac{1}{1} + \frac{1}{1} + \frac{1}{1}) = 10$

$U_{md}(R_5) = 10$ (as shown in Example 1), so $U_{md}(R_5)$ minimizes U_{md} for equilateral 5-gons. □

This method cannot be used to prove that regular n-gons minimize for all equilateral n-gons since we can no longer use Theorem 3 to gain an understanding of the minimum distance between each non-adjacent pair of edges when $n > 5$.

4. FUTURE PLANS

Cantarella et. al. [1] are able to prove that Möbius Energy is minimized for a standard planar circle by making a new version of Theorem 15 for a function $F(x, y)$ which is convex and decreasing[1]. If we could similarly alter Theorem 14 then we could prove at least the equilateral case by having $g(t) = \frac{1}{t}$.

We now provide an explanation of work done by Cantarella et. al. in order to show some ways in which Lükő's results can be manipulated to apply to convex and decreasing functions.

Definition 11. Let c be a closed, unit-speed curve in $\mathbb{R}^n$ with length 2π. Let $\lambda(s)$ be the length of a chord with arclength s on the unit circle.

Theorem 17 ([1]'s version of Lükő's)**.** *If $g(t)$ is increasing and concave on $(0, |0 - s|^2]$, then*

$$\frac{1}{2\pi}\int g\left(|x(t+s) - x(t)|^2\right) dt \leq g(\lambda^2(s))$$

with equality if and only if c is the unit circle.

Note that for $g(t)$ this theorem is not strong enough to prove that Möbius energy is minimized. However we can produce the following inequality which looks vaguely like the continuous version of (4), namely:

$$\frac{1}{\frac{1}{2\pi}\int |x(t+s) - x(t)|^2 dt} \geq \frac{1}{\lambda^2(s)}$$

The way in which Cantarella et. al. are able to resolve this problem is by applying a "Wirtinger type inequality" in order to rederive Lükő's results [1] to apply to functions from $\mathbb{R}^2 \to \mathbb{R}$ which are convex and decreasing. This allows them to create the following theorem, which allowed them to prove that Möbius energy is minimized for planar circles.

Theorem 18 ([1])**.** *If $F(x, y) : \mathbb{R}^2 \to \mathbb{R}$, is convex and decreasing in x^2 for $x^2 \in (0, y^2]$ and $y \in (0, \pi)$ then*

$$\iint F(|x(t) - x(s)|, |s - t|)\, ds\, dt$$

is uniquely minimized on the unit circle.

Remark 4. Note that for $F(x, y) = \frac{1}{x^2} - \frac{1}{y^2}$, Theorem 18 implies that Möbius energy is uniquely minimized on the unit circle.

Trying to rework the discrete version of Theorem 13 may be possible by similarly finding a discrete Wirtinger-type inequality and making a version of Theorem 14 which applies to a function which is convex and decreasing. This might allow us to prove the conjecture for all equilateral n-gons. This would be a significant step towards proving the conjecture. It will then remain to be shown that equilateral n-gons minimize U_{md} for all n-gons. It might also be possible to have a similar inequality for a function of the form $F(x, y)$ where the output is the minimum distance energy of a pair of edges. This would allow us to prove our conjecture for all n-gons, both equilateral and non-equilateral. The following techniques may also help in continuing.

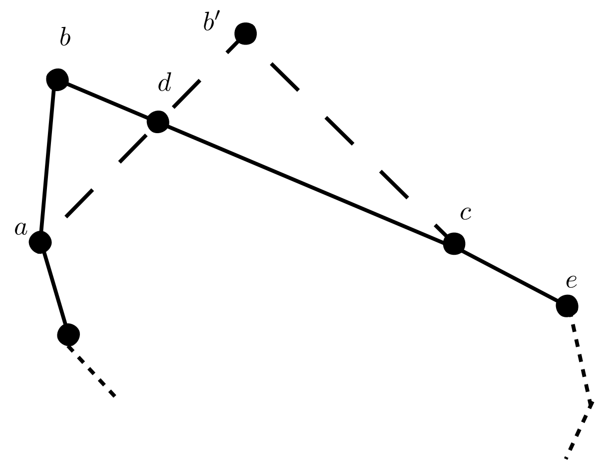

FIG. 10: Changing edge lengths can change convexity.

4.1. n-gons, not necessarily equilateral

It may be possible to show that equilateral n-gons minimize U_{md} by changing pairs of edge lengths. However, experiments with this method have shown that sometimes changing edge lengths has caused the minimum distance between a pair of edges to be non-convex (implying that U_{md} will not necessarily be minimimum by Theorem 10).

Another possible method involves proving that polygons which can be inscribed on a circle have least U_{md}, and then using a method called *midpoint stretching*, to obtain a regular (and therefore equilateral and equiangular) n-gon.

4.1.1. Changing edge lengths

Let $\overline{ab}$ and $\overline{bc}$, be consecutive edges with $\ell(\overline{ab}) \neq \ell(\overline{bc})$. Let b be a point on an ellipse with foci at a and c. Using properties of ellipses, it is easy to find a point on the ellipse b', such that $\ell(\overline{ab'}) = \ell(\overline{b'c})$. The area of $\triangle abc$ is less than that of the new $\triangle ab'c$ and thus equalizing the length produces a polygon with greater area [2]. Following the connection that has been made between area as giving a sense of the average minimum distance across an n-gon this seems hopeful.

However, there can be situations where changing the lengths of edges can force a convex polygon to become non-convex. Figure 10 gives an example. If we allow for a combination of edge length changes and flips, however, this will no longer be the case, and it seems likely that such changes of edge length, overall, would decrease in U'_{md}.

4.1.2. Cylic polygons and midpoint stretching

Definition 12 ([9]). Let θ_i denote the measure of a central angle, a_i a corresponding chord length. $\theta_i = 2\arcsin(\frac{a_i/2}{r}) = 2\arcsin\left(\frac{k(r)a_i}{2}\right)$ where $k(r) := \frac{1}{r}$.

Theorem 19 ([9]). *If $a_1 + a_2 + \ldots a_{n-1} > a_n$ then there exists a covex cyclic polygon with edge lengths $(a_1 \ldots a_n)$ which is unique up to isometry.*

It is known that a polygon inscribed in a circle has a greater area that any other polygon with the same edge lengths [2, 10]. Polygons which can be inscribed on a circle are known as "chordal" or "cyclic" polygons [9]. A regular n-gon is cyclic, so perhaps investigating if cyclic polygons have minimized U'_{md} could give more evidence towards our conjecture. I. Pinelis [9] proves that for a set of edge lengths, their exists a unique cyclic polygon with those edge lengths. Finding the radius of a cyclic polygon for a set of edge lengths can also give us something to use to compare with $U_{md}(R_n)$. For non-adjacent edges X and Y maximum $md(X, Y)$ will be the diameter of the circle. However, the necessary task for implementing Theorem 19 is to create an algorithm for inscribing edges on a circle that also allows us to track changes in the minimum distance between edges.

The circle is also valuable in determining minimum distance, since chord length can often be used to relate distances on a polygon inscribed on a circle. This may simplify the problem since equilateral polygons inscribed on a circle must also be equiangular.

Fortunately, there exists a great deal of published research on cyclic polygons. This material gives us several possible algorithms for equalizing edge lengths. L. Hitt and X. Zhang [5] study sequences of "midpoint-stretching polygons." They take a chordal polygon, and create another chordal polygon with vertices at the midpoints of all edges on the first polygon. They find that if this process is repeated it creates a sequence of polygons that will converge to a regular n-gon. The problem that occurs, however, when one is trying to apply this move to study U_{md}, is that it is hard to generalize where the point of minimum distance will occur on an edge.

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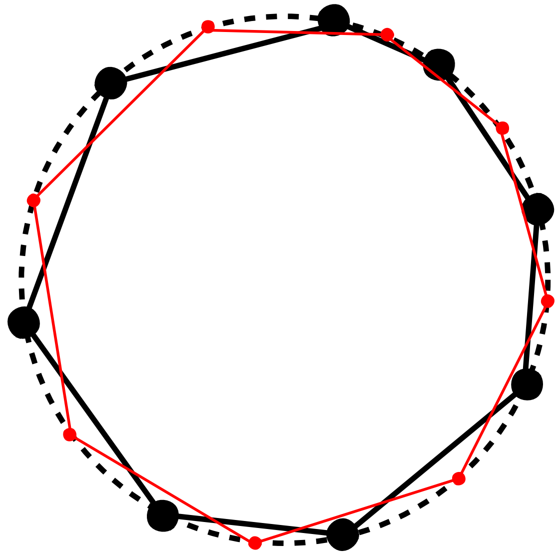

FIG. 11: The polygon shown in red is the result of a mid-point stretching of the polygon shown in black.

who can from time to time be seen in the Smith College Math Forum for creating an exciting and rich academic environment.

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[1] A. Abrams, J. Cantarella, J. H. G. Fu, M. Ghomi and R. Howard, Circles minimize most knot energies, *Topology* **42** (2003) 381–394.

[2] R. V. Benson, *Euclidean Geometry and Convexity*, (McGraw-Hill, 1966).

[3] R. F. Demar. A simple approach to isoperimetric problems in the plane, *Mathematics Magazine* **48**:1 (1975) 1–12.

[4] P. Erdös, Problem number 3763, *Amer. Math. Monthly* **42** (1935) 627.

[5] L. R. Hitt and X. Zhang, Dynamic geometry of polygons, *Elem. Math.* **56** (2001) 21–37.

[6] G. Lükő, On the mean length of the chords of a closed curve, *Israel J. Math.* **4** (1966) 23–32.

[7] L.A. Lyusternik, Convex Figures and Polyhedra, (Boston: D.C. Heath & Company, 1966).

[8] I. Niven, *Maxima and Minima Without Calculus*, (Mathematical Association of America, 1981).

[9] I. Pinelis, Cyclic polygons with given edge lenths: Existence and uniqueness, *J. Geom. 82* (2005) 156–171.

[10] G. Polya, *Induction and Analogy in Mathematics*, Princeton University Press, 1973.

[11] E. J. Rawdon and J. K. Simon, Polygonal approximation and energy of smooth knots, *J. Knot Theory Ramifications* **15**:4 (2006), 429–451.

[12] E. J. Rawdon and J. Worthington. Error Analysis of the Minimum Distance Energy of a Polygonal Knot and the Möbius Energy of an Approximating Curve. (2005), *Preprint*.

[13] G. T. Sallee, Stretching chords of space curves, *Geometriae Dedicata* **2** (1973) 311–315.

[14] J. Simon, Energy functions for polygonal knots, *J. Knot Theory Ramifications* **3**:3 (1994), 299–320.

[15] R. Speller, Convexity and minimum distance energy, REU Project, California State University San Bernardino (CSUSB), 2007.

[16] J. Tam, The minimum distance energy for polygonal unknots, REU Project, California State University San Bernardino (CSUSB), 2006.

[17] G. Toussaint, The Erdös-Nagy theorem and its ramifications, *Computational Geometry* **31** (2005) 219–236.

[18] Y. Wu, MING, a computer program used to minimize the minimum distance energy of and visualize polygonal knots, University of Iowa http://www.math.uiowa.edu/%7Ewu/.

[19] I. M. Yaglom and V. G. Boltyanskiĭ, *Convex Figures*, (Holt, Rinehart and Winston, 1961).